AGNAGN seminar
chapter 6. Internal Dynamics of Gaseous Nebulae 6.1-6.5

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6.1 Introduction

So far, we have discussed static gaseous nebulae.
In this chapter we utilize hydrodynamics and describe time evolution (mainly on expansion).

- matter-bounded: AGN, Novae (expand into vacuum)
- ionization-bounded: HII region (expand by ionizing source)

Contents

- Basics of hydrodynamics and ionization equation (time-dependent)
- Free expansion into a vacuum
- Shocks
- Ionization fronts
- Dynamics of expanding HII region

6.2 Hydrodynamic equations of motion

- **Lagrangian** reference frame: The frame which moves with the fluid element.
- **Eulerian** reference frame: The frame which is stationary and describe the fluid at fixed point.

For any physical quantity f, the time derivative in Lagrangian frame D/Dt is

$$\frac{Df}{Dt} = \underbrace{\frac{\partial f}{\partial t}}_{\text{at fixed point}} + \underbrace{\boldsymbol{u} \cdot \nabla f}_{\text{advection}} \tag{1}$$

Advection : the changes due to the flow of material into the region.
In 1D system, it becomes simple

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} \tag{2}$$

6.2 Momentum equation

Let's consider the equation of motion in a hydrodynamic system like nebulae. The equation of motion in the Lagrangian frame is

$$\rho \frac{D\boldsymbol{u}}{Dt} = \text{(forces to the element)} \tag{3}$$

Possible forces

- pressure gradient $-\nabla P$
- gravity $-\rho \nabla \phi$ (ϕ:the gravitational potential)
- viscous $\nabla \cdot \sigma$ ($(\sigma)_{ij} = \sigma_{ij}$: viscous stress tensor)
- Lorentz force $q(\boldsymbol{u} \times \boldsymbol{B})$ (q: charge density)

6.2 Momentum equation

In many cases, viscous and Lorentz forces are neglected. So,

momentum equation

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla P - \rho \nabla \phi \tag{4}$$

For the case of 1D flow,

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial P}{\partial x} - \rho \frac{\partial \phi}{\partial x} \tag{5}$$

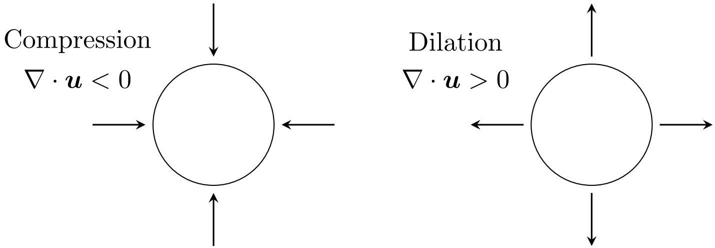
Note that in the cases of nebulae with strong magnetic field $\boldsymbol{B}$, the omission of Lorentz force $q(\boldsymbol{u} \times \boldsymbol{B})$ is incorrect.

6.2 Equation of continuity

mass conservation(equation of continuity)

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \boldsymbol{u} \tag{6}$$

RHS of this equation reflects the effect of compression (or dilation).



Hereafter, we call terms $\propto -\nabla \cdot \boldsymbol{u}$: **"dilation term"**.

6.2 Equation of continuity

In 1D system,

$$\frac{D\rho}{Dt} = -\rho\frac{\partial u}{\partial x} \tag{7}$$

and we can rewrite it in the Eulerian frame

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = -\rho\frac{\partial u}{\partial x} \tag{8}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad \text{(equation of continuity)} \tag{9}$$

You might be more familiar with this expression than eq(6).

6.2 Energy equation

In the case of static nebulae, the thermal balance is like

$$G = L_R + L_{FF} + L_C \tag{10}$$

where G : energy input by central star, L: energy loss by recombination, free-free emission, emission lines.
We can generalize this to non-static nebulae.
According to **the 1st law of thermodynamics**,

$$dU = \bar{d}Q + \bar{d}W \tag{11}$$
$$VdU = V(G - L)dt - PdV \tag{12}$$

Note that $(G - L)dt$ means the net heating. And the internal energy (per unit volume) can be expressed like

$$U = \sum_j \frac{3}{2} n_j k_\mathrm{B} T \tag{13}$$

6.2 Energy equation

Divide $VdU = V(G-L)dt - PdV$ by Vdt and depict in Lagrange frame,

$$\frac{DU}{Dt} = (G-L) - P\nabla\cdot\boldsymbol{u} - \underbrace{U\nabla\cdot\boldsymbol{u}}_{\text{dilation}} \tag{14}$$

here we use $\frac{\mathrm{d}V}{\mathrm{d}t} = V\nabla\cdot\boldsymbol{u}$. The last term on RHS $-U\nabla\cdot\boldsymbol{u}$ reflects the dilation effect (analogy to eq(6))

Using mass conservation $\frac{D\rho}{Dt} = -\rho\nabla\cdot\boldsymbol{u}$, the work term becomes

$$-P\nabla\cdot\boldsymbol{u} = -P\times\left(-\frac{1}{\rho}\frac{D\rho}{Dt}\right) = \frac{P}{\rho}\frac{D\rho}{Dt} \tag{15}$$

and then we get

energy equation

$$\frac{DU}{Dt} = \underbrace{(G-L)}_{\text{heating}} + \underbrace{\frac{P}{\rho}\frac{D\rho}{Dt}}_{\text{work}} - \underbrace{U\nabla\cdot\boldsymbol{u}}_{\text{dilation}} \tag{16}$$

6.2 Energy equation

In 1D system,

$$\frac{DU}{Dt} = (G - L) + \frac{P}{\rho}\frac{D\rho}{Dt} - U\frac{\partial u}{\partial x} \tag{17}$$

$$\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} = (G - L) + \frac{P}{\rho}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right) - U\frac{\partial u}{\partial x} \tag{18}$$

$$\frac{\partial U}{\partial t} + \frac{\partial (Uu)}{\partial x} = (G - L) + \frac{P}{\rho}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right) \tag{19}$$

- LHS of this expression corresponds to that of equation of continuity $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$.
- But U is not conserved, so RHS(heating and work) exists.

6.2 More fundamental form of energy equation

It is somewhat important to consider the time evolution of the internal energy per unit mass $E = U/\rho$.

$$\frac{DE}{Dt} = \frac{D}{Dt}\left(\frac{U}{\rho}\right) = \frac{1}{\rho}\frac{DU}{Dt} + U\frac{D}{Dt}\left(\frac{1}{\rho}\right) \quad \text{(Leibniz rule)} \tag{20}$$

Substitute energy equation(16): $\dfrac{DU}{Dt} = (G - L) + \dfrac{P}{\rho}\dfrac{D\rho}{Dt} - U\nabla\cdot\boldsymbol{u}$

$$\frac{DE}{Dt} = \frac{G-L}{\rho} + \frac{P}{\rho^2}\frac{D\rho}{Dt} - \frac{U}{\rho}\nabla\cdot\boldsymbol{u} + U\frac{D}{Dt}\left(\frac{1}{\rho}\right) \tag{21}$$

$$= \frac{G-L}{\rho} - P\frac{D}{Dt}\left(\frac{1}{\rho}\right) - \frac{U}{\rho^2}\left[\rho\nabla\cdot\boldsymbol{u} + \frac{D\rho}{Dt}\right] \tag{22}$$

$$= \frac{G-L}{\rho} - P\frac{D}{Dt}\left(\frac{1}{\rho}\right) \quad (\because (6)) \tag{23}$$

6.2 More fundamental form of energy equation

energy equation(fundamental form)

$$\frac{DE}{Dt} = \frac{G-L}{\rho} - P\frac{D}{Dt}\left(\frac{1}{\rho}\right) \tag{24}$$

We can derive this **directly(intuitively)** from the 1st law of thermodynamics. Let's consider a certain region with volume V, mass M and its subset with volume V/M.

$$dU = \bar{d}Q + \bar{d}W \tag{25}$$

$$d(VU) = (G-L)V\,dt - PdV \quad \text{(whole region)} \tag{26}$$

$$d\left(\frac{VU}{M}\right) = \frac{(G-L)V\,dt}{M} - Pd\left(\frac{V}{M}\right) \quad \text{(subset)} \tag{27}$$

divide both sides by dt and introduce $M/V = \rho$, then we get

$$\frac{dE}{dt} = \frac{G-L}{\rho} - P\frac{d}{dt}\left(\frac{1}{\rho}\right) \tag{28}$$

6.2 Comments & Quiz on the energy equations

We have explained 2 forms of the energy equation.

- eq(24) can be derived from the 1st law and the replacement $d/dt \to D/Dt$.
- We derived eq(24) without using the equation of continuity.
- Therefore, it would be better(reasonable) to derive eq(16) from eq(24).

Quiz!

The energy equation of U (eq(16)) has the dilation term $-U\nabla \cdot \boldsymbol{u}$ on its RHS. However, the fundamental form (eq(24)) doesn't have a term like $-E\nabla \cdot \boldsymbol{u}$. **Why ?**

6.2 Intro to ionization equation

Here we denote any element :X , and its i-times ionized stage X^{+i}.
According to Chapter 2 , ionization equilibrium between X^{+i} and X^{+i+1} is

$$(\text{photoionization}) = (\text{recombination}) \tag{29}$$

$$\underbrace{n(X^{+i}) \int_{\nu_i}^{\infty} \frac{4\pi J_\nu}{h\nu} a_\nu(X^{+i}) d\nu}_{X^{+i} \to X^{+i+1}} = \underbrace{n(X^{+i+1}) n_e \alpha(X^{+i}, T)}_{X^{+i+1} \to X^{+i}} \tag{30}$$

where,
$a_\nu(X^{+i})$: photoionization cross section,
$\alpha(X^{+i}, T)$: recombination cross section.
Then, we **generalize** this equation to **non-static system**.

6.2 Ionization equation

Here we consider the time evolution of $n(X^{+i})$. This time, we consider not just X^{+i}, X^{+i+1}, but also X^{+i-1}.

Ionization equation

$$\frac{Dn(X^{+i})}{Dt} = -\,\mathrm{pho}(X^{+i}) + \mathrm{rec}(X^{+i+1}) - \mathrm{rec}(X^{+i}) + \mathrm{pho}(X^{+i-1}) - n(X^{+i})\nabla\cdot\boldsymbol{u} \quad (31)$$

where

$$\mathrm{pho}(X^{i}) = n(X^{i})\int_{\nu_i}^{\infty}\frac{4\pi J_\nu}{h\nu}a_\nu(X^{i})d\nu \quad (32)$$

$$\mathrm{rec}(X^{i}) = n(X^{i})n_e\alpha_{\mathrm{A}}(X^{i-1},T) \quad (33)$$

Don't forget the dilation term $-n(X^{+i})\nabla\cdot\boldsymbol{u}$.

Expanding HII region.
Substitute $X = \mathrm{H}$, $i = 0$ to the ionization equation.

$$\frac{Dn(\mathrm{H}^0)}{Dt} = \frac{\partial n(\mathrm{H}^0)}{\partial t} + \boldsymbol{u} \cdot \nabla n(\mathrm{H}^0) \tag{34}$$

$$= -\mathrm{pho}(\mathrm{H}^0) + \mathrm{rec}(\mathrm{H}^+) - n(\mathrm{H}^0)\nabla \cdot \boldsymbol{u} \tag{35}$$

$$\frac{\partial n(\mathrm{H}^0)}{\partial t} = -\mathrm{pho}(\mathrm{H}^0) + \mathrm{rec}(\mathrm{H}^+) - \nabla \cdot [n(\mathrm{H}^0)\boldsymbol{u}] \tag{36}$$

Integrate over whole HII region

$$\int \frac{\partial n(\mathrm{H}^0)}{\partial t} dV = -\int \mathrm{pho}(\mathrm{H}^0) dV + \int \mathrm{rec}(\mathrm{H}^+) dV - \int \nabla \cdot [n(\mathrm{H}^0)\boldsymbol{u}] dV \tag{37}$$

In HII region, time scale of photoionization and recombination is much shorter than dynamical time. So, we assume $\frac{\partial n(\mathrm{H}^0)}{\partial t} = 0$,

$$\int n(\mathrm{H}^0) \int_{\nu_0}^{\infty} \frac{4\pi J_\nu}{h\nu} a_\nu(\mathrm{H}^0) d\nu dV = \int n_p n_e \alpha_{\mathrm{A}}(\mathrm{H}^0, T) dV - \int \nabla \cdot [n(\mathrm{H}^0)\boldsymbol{u}] dV \tag{38}$$

6.2 Application:Expanding HII region

We assume on-the-spot approximation ($J_\nu \to J_{\nu,\text{stellar}}$ and $\alpha_\text{A} \to \alpha_\text{B}$).

$$\Phi(\text{H}^0) = \int n_p n_e \alpha_\text{B}(\text{H}^0, T) dV - \int \nabla \cdot [n(\text{H}^0)\boldsymbol{u}] dV \tag{39}$$

where $\Phi(\text{H}^0)$ is the total injection of ionizing photons from the central star.

$$\Phi(\text{H}^0) := \int n(\text{H}^0) \int \frac{4\pi J_{\nu,\text{stellar}}}{h\nu} a_\nu(\text{H}^0, T) d\nu dV \tag{40}$$

Second term in RHS,

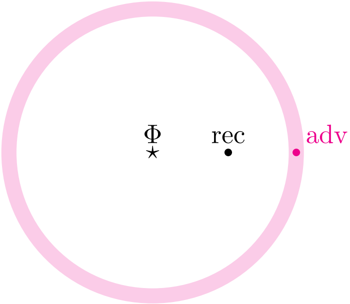
$$\int \nabla \cdot [n(\text{H}^0)\boldsymbol{u}] dV = \int_S n(\text{H}^0)\boldsymbol{u} \cdot d\boldsymbol{S}, \quad \text{(S: front of HII region)} \tag{41}$$

6.2 Application:Expanding HII region

The final result of the ionization equation of HII region is

$$\Phi(\mathrm{H}^0) = \underbrace{\int n_p n_e \alpha_{\mathrm{B}}(\mathrm{H}^0, T) dV}_{\text{recombination}} - \underbrace{\int_S n(\mathrm{H}^0) \boldsymbol{u} \cdot d\boldsymbol{S}}_{\text{advection}} \tag{42}$$

Drawing is



Typical value

- $\Phi(\mathrm{H}^0) = 2 \times 10^9 \mathrm{cm}^{-2}\,\mathrm{s}^{-1}$
- $n = 10\mathrm{cm}^{-3}$
- $u = 20\mathrm{km\,s}^{-1}$

Then we get

$$\frac{\text{Advection}}{\Phi(\mathrm{H}^0)} \simeq 0.01 \tag{43}$$

Therefore, at the front of HII region, the luminosity of the recombination lines decrease about 1% because of the advection.

6.2 Caution!

Equation (6.13) in the textbook is somewhat confusing.

$$\Phi(\mathrm{H}^0) = \int [n_e n_p \alpha_\mathrm{B} + \boldsymbol{u} \cdot \nabla n] dr \simeq \int n_e n_p \alpha_\mathrm{B} dr + n(\mathrm{H}^0) u \tag{6.13}$$

First term is same as our result(recombination term), but the sign of the second term is different. This is because the viewpoints are different.
Our advection term on 1D sytem is written like

$$-\int_S n(\mathrm{H}^0) \boldsymbol{u} \cdot d\boldsymbol{S} = -n(\mathrm{H}^0) u_r =: n(\mathrm{H}^0) u_r' \tag{44}$$

u_r is the velocity toward the outside, and u_r' is the velocity into the inside.
Textbook adopts the viewpoint at which H^0 outside the HII region come into it and sets u as u_r' because it becomes positive when HII region is expanding.

6.2 Equation of state

equation of state

$$P = \frac{\rho k T}{\mu m_{\mathrm{H}}} = n_{\mathrm{total}} k T \tag{45}$$

Simple limiting cases,

- Isothermal: In HII region, the balance between heating and cooling determine T.
- Adiabatic:

$$P = K \rho^{\gamma} \tag{46}$$

6.2 Time-Steady Limit

Time-Steady limit $\partial/\partial t = 0$.
Equation of continuity becomes

$$\frac{\partial(\rho u)}{\partial x} = 0 \quad \rightarrow \quad \Phi := \rho u \text{ is conserved} \tag{47}$$

Momentum equation (ignoring gravity term)

$$\rho u \frac{\partial u}{\partial x} = -\frac{\partial P}{\partial x} \tag{48}$$

Using

$$\frac{\partial(\rho u^2)}{\partial x} = 2\rho u \frac{\partial u}{\partial x} + u^2 \frac{\partial \rho}{\partial x} \tag{49}$$

$$= 2\rho u \frac{\partial u}{\partial x} - u\rho \frac{\partial u}{\partial x} = \rho u \frac{\partial u}{\partial x} \quad (\because \text{ eq of continuity}) \tag{50}$$

then

$$\frac{\partial(\rho u^2)}{\partial x} = -\frac{\partial P}{\partial x} \quad \rightarrow \quad \Pi = P + \rho u^2 = \rho c^2 + \rho u^2 \tag{51}$$

where $c = (\gamma kT/\mu m_{\rm H})^{1/2}$ is sound speed.

6.2 Time-Steady Limit

1D momentum equation (gravity is ignored)

$$\rho \frac{Du}{Dt} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial P}{\partial x} \tag{52}$$

If the system is adiabatic ($P = K\rho^\gamma$), time-steady limit is

$$\rho u\frac{\partial u}{\partial x} = -\frac{\partial P}{\partial x} = -K\gamma\rho^{\gamma-1}\frac{\partial \rho}{\partial x} \tag{53}$$

$$u\frac{\partial u}{\partial x} = -K\gamma\rho^{\gamma-2}\frac{\partial \rho}{\partial x} = -\frac{\gamma}{\gamma-1}\frac{\partial(K\rho^{\gamma-1})}{\partial x} \tag{54}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = -\frac{\gamma}{\gamma-1}\frac{\partial}{\partial x}\left(\frac{P}{\rho}\right) \tag{55}$$

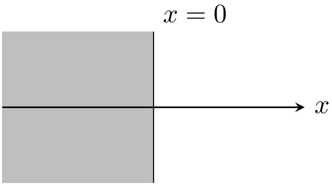
Therefore,

$$H = \underbrace{\frac{1}{2}u^2}_{\text{kinetic}} + \underbrace{\frac{\gamma}{\gamma-1}\frac{P}{\rho}}_{\text{enthalpy}} \tag{56}$$

is conserved. This is the alternative to the energy equation.

6.3 Free expansion into a vacuum

Set up: $u = 0, \rho = \rho_0, P = P_0$ in $x < 0$ and vacuum in $x > 0$.
At $t = 0$, the wall $x = 0$ is removed and gas in $x < 0$ starts to expand into $x > 0$.



Mass and momentum equations

$$\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \tag{57}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\partial P}{\partial x} = 0 \tag{58}$$

Also, we assume ideal adiabatic gas

$$P = K\rho^\gamma, \quad c^2 = \frac{dP}{d\rho} = \gamma\frac{P}{\rho} \tag{59}$$

6.3 Derivation of Riemann invariant

$$2cdc = d\left(\gamma \frac{P}{\rho}\right) = \frac{\gamma}{\rho} dP - \frac{\gamma P}{\rho^2} d\rho = \frac{\gamma}{\rho} c^2 d\rho - \frac{c^2}{\rho} d\rho = c^2(\gamma - 1)\frac{d\rho}{\rho} \quad (60)$$

$$\therefore \quad \frac{d\rho}{\rho} = \frac{2}{\gamma - 1}\frac{dc}{c} \quad (61)$$

Substitute this relation for mass conservation.

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} + \frac{u}{\rho}\frac{\partial \rho}{\partial x} = 0 \quad (62)$$

$$\frac{\partial}{\partial t}\left(\frac{2}{\gamma - 1}c\right) + c\frac{\partial u}{\partial x} + u\frac{\partial}{\partial x}\left(\frac{2}{\gamma - 1}c\right) = 0 \quad (63)$$

Also substitute eq(61) for momentum conservation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x} = -c\frac{\partial}{\partial x}\left(\frac{2}{\gamma - 1}c\right) \quad (64)$$

6.3 Riemann invariant

Calculate eq(63) $\pm$ eq(64)

$$\frac{\partial}{\partial t}\left(u \pm \frac{2}{\gamma - 1}c\right) + (u \pm c)\frac{\partial}{\partial x}\left(u \pm \frac{2}{\gamma - 1}c\right) = 0 \tag{65}$$

Then we define $J_{\pm}$ (Riemann invariant)

$$J_{\pm} := u \pm \frac{2}{\gamma - 1}c, \quad \frac{\partial J_{\pm}}{\partial t} + (u \pm c)\frac{\partial J_{\pm}}{\partial x} = 0 \tag{66}$$

J_+ is transported with velocity $u + c$ toward $x > 0$ and conserved with the motion.
Therefore,

$$J_+ = u + \frac{2}{\gamma - 1}c = 0 + \frac{2}{\gamma - 1}c_0 \quad \rightarrow \quad u = \frac{2}{\gamma - 1}(c_0 - c) \tag{67}$$

At the expanding edge (front), $c = 0$ is satisfied ($\because$ it contacts with vacuum).

$$(\text{front speed}): \quad u_e = \frac{2}{\gamma - 1}c_0 \tag{68}$$

6.3 Free expansion into a Vacuum

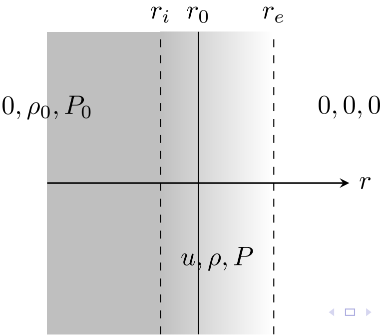
Cloud freely expands into a Vacuum : Novae, AGNs
In spherical system, just replace $x \to r$.
At later time t, the outer edge has reached r_e

$$r_e = r_0 + u_e t \tag{69}$$

While the rarefaction wave has reached r_i

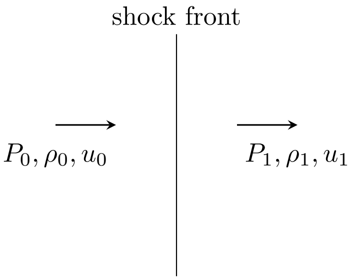
$$r_i = r_0 - ct \tag{70}$$

Velocity increases from 0 (at r_i) to u_e (at r_e).



6.4 Shocks

In a reference frame moving with the shock front.



Conservations are

$$\rho_0 u_0 = \rho_1 u_1 \tag{71}$$

$$P_0 + \rho_0 u_0^2 = P_1 + \rho_1 u_1^2 \tag{72}$$

$$\frac{1}{2}u_0^2 + \frac{\gamma}{\gamma - 1}\frac{P_0}{\rho_0} = \frac{1}{2}u_1^2 + \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1} \tag{73}$$

6.4 Rankine—Hugoniot relation

Introduce Mach number M

$$M = \frac{|u_0|}{c_0}, \quad c_0 = \sqrt{\frac{\gamma P_0}{\rho_0}} = \sqrt{\frac{\gamma k T_0}{\mu m_{\mathrm{H}}}} : \text{sound speed} \tag{74}$$

Using the conserved quantities, we can express the ratios $P_1/P_0, \rho_1/\rho_0$ with only M.

$$\frac{P_1}{P_0} = \frac{2\gamma}{\gamma+1} M^2 - \frac{\gamma-1}{\gamma+1} \tag{75}$$

$$\frac{\rho_1}{\rho_0} = \frac{(\gamma+1)M^2}{(\gamma-1)M^2+2} \tag{76}$$

Strong shock limit $M \to \infty$, $\dfrac{\rho_1}{\rho_0} \to \dfrac{\gamma+1}{\gamma-1} = 4$(for adiabatic case).

6.4 Isothermal shock (zero-order approximation)

Typical gaseous nebula or HII region:

$$\text{(radiative time scale)} \ll \text{(dilation time scale)} \tag{77}$$

Therefore, to a zero-order approximation,

- Region just behind the shock front has high temperature ($T_0 \to T$).
- But immediately cooled by radiation and return to the original temperature ($T \to T_0$).
- So, we can regard these systems as isothermal.

Mathematically, the ratio $\frac{\rho_1}{\rho_0}$ can be obtained by just taking the limit $\gamma \to 1$(isothermal:$P = nkT$)

$$\frac{\rho_1}{\rho_0} = M^2 \tag{78}$$

6.5 ionizing photon flux

Ionization front: not only ρ, u and P, but also the degree of ionization change discontinuously.
Here we introduce basic concept $\phi(\mathrm{H}^0)$.
Mass flux can be expressed with ionizing photon flux ($\phi_i = \phi(\mathrm{H}^0)$):

$$\rho_0 u_0 = \rho_1 u_1 = m_i \phi_i \tag{79}$$

where

$$\phi(\mathrm{H}^0) = \frac{Q(\mathrm{H}^0)}{4\pi r^2} = \int_{\nu_0}^{\infty} \frac{\pi F_\nu}{h\nu} d\nu, \quad (\pi F_\nu : \text{energy flux from star}) \tag{80}$$

and m_i: mass of electron-ion pair.

6.5 Expanding ionization front

We now formulate the dynamics of the expanding ionization front.
We assume that the advection term in eq(42) is negligible and inside the front is completely ionized.
Totale HII number inside the front is simply given like

$$N(t) = \frac{4\pi}{3} r^3(t) n, \quad r(t) : \text{ionization front} \tag{81}$$

And its time evolution is

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \underbrace{Q(\mathrm{H}^0)}_{\text{ionization}} - \underbrace{\int_V n_e n_p \alpha_{\mathrm{B}} dV}_{\text{recombination}} \tag{82}$$

Since $n_e \simeq n_p =: n$, recombination term is approximately $n^2 \alpha_{\mathrm{B}} \frac{4\pi}{3} r^3(t)$.
Resulting time evolution is

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{4\pi}{3} r^3 n \right) = Q(\mathrm{H}^0) - n^2 \alpha_{\mathrm{B}} \frac{4\pi}{3} r^3 \tag{83}$$

6.5 Expanding ionization front

$$4\pi r^2 n \frac{\mathrm{d}r}{\mathrm{d}t} = Q(\mathrm{H}^0) - n^2 \alpha_\mathrm{B} \frac{4\pi}{3} r^3 \tag{84}$$

$$u = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{Q(\mathrm{H}^0)}{4\pi r^2 n} - \frac{\alpha_\mathrm{B} n r}{3} \tag{85}$$

This is the speed of the ionization front.
Solution $r(t)$ can analytically calculated by multiplying both sides by $3r^2$

$$\frac{\mathrm{d}}{\mathrm{d}t} r^3 = \frac{3Q(\mathrm{H}^0)}{4\pi n} - \alpha_\mathrm{B} n r^3 \tag{86}$$

This is one of the most famous ODEs and its solution is given like

$$r^3(t) = \frac{3Q(\mathrm{H}^0)}{4\pi \alpha_\mathrm{B} n^2} [1 - \exp(-\alpha_\mathrm{B} n t)] \tag{87}$$

Caution! : In the textbook , 4π doesn't appear. It may be mistake.

6.5 Excess energy: Alternative to the energy eq

Introduction of the excess kinetic energy (per unit mass) $q^2/2$

$$\frac{1}{2}m_i q^2 = \frac{1}{\phi_i}\int_{\nu_0}^{\infty}\frac{\pi F_\nu}{h\nu}(h\nu - h\nu_0)d\nu \tag{88}$$

Using $q^2/2$, the alternative to the energy equation is

$$\frac{1}{2}u_0^2 + \frac{\gamma}{\gamma-1}\frac{P_0}{\rho_0} + \frac{1}{2}q^2 = \frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} \tag{89}$$

$\frac{1}{2}q^2$ on LHS: the excess energy by photoionization.
cf) eq(73)

6.5 1st-order correction for Isothermal shock

Because of the additional term $\frac{1}{2}q^2$, the equilibrium temperature behind the front differs from that ahead of the front.

$$\frac{P_0}{\rho_0} = \frac{kT_0}{\mu_0 m_{\rm H}}, \quad \frac{P_1}{\rho_1} = \frac{kT_1}{\mu_1 m_{\rm H}} \tag{90}$$

where

$$T_0 \ll T_1, \quad \mu_0 \simeq 1, \mu_1 \simeq \frac{1}{2} \tag{91}$$

Resulting corrected-Rankine-Hugoniot relation

$$\frac{\rho_1}{\rho_0} = \frac{c_0^2 + u_0^2 \pm [(c_0^2 + u_0^2)^2 - 4c_1^2 u_0^2]^{1/2}}{2c_1^2} \tag{92}$$

6.5 R and D critical front

Physically $\dfrac{\rho_1}{\rho_0}$ must be real $((c_0^2+u_0^2)^2-4c_1^2u_0^2\geq 0)$. Therefore,

$$u_0 \geq c_1 + \sqrt{c_1^2 - c_0^2} =: u_R \simeq 2c_1 \tag{93}$$

or

$$u_0 \leq c_1 - \sqrt{c_1^2 - c_0^2} =: u_D \simeq \frac{c_0^2}{2c_1} \tag{94}$$

Subscripts R, D stand for "rare" and "dense" respectively.

- R front: Rare gas move supersonically ($u_R \gg c_0$) into H^0 (ahead of the front).
- D front: Dense gas move subsonically ($u_0 < u_D < c_0$).

6.5 R-type front

If a hot star were instantaneously " turned on " in an infinite homogeneous H^0 cloud, a fast R front would run at first. So, we assume $u_0 \gg c_1 (\gg c_0)$.

$$\frac{\rho_1}{\rho_0} \simeq \frac{u_0^2 \pm [u_0^4 - 4c_1^2 u_0^2]^{1/2}}{2c_1^2} \quad (\because c_0^2 \simeq 0) \tag{95}$$

$$\simeq \frac{u_0^2}{2c_1^2}\left[1 \pm \left(1 - 2\frac{c_1^2}{u_0^2} - \frac{c_1^4}{u_0^4}\right)\right] \tag{96}$$

$$\simeq \begin{cases} \dfrac{u_0^2}{c_1^2}\left(1 - \dfrac{c_1^2}{u_0^2}\right) \gg 1 & \text{(Strong)} \\ 1 + \dfrac{c_1^2}{2u_0^2} \simeq 1 & \text{(Weak)} \end{cases} \tag{97}$$

Behind the strong R front, the density is becomes much larger (strongly compressed).

Using mass conservation $\rho_0 u_0 = \rho_1 u_1$, we calculate u_1

$$u_1 = \frac{\rho_0}{\rho_1} u_0 = \begin{cases} \dfrac{c_1^2}{u_0} \ll c_1 & \text{(Strong)} \\ \\ u_0 \gg c_1 & \text{(Weak)} \end{cases} \tag{98}$$

Therefore, the velocity behind the strong/weak R front u_1 is subsonic/supersonic.

However, the strong R front cannot exist in nature because

- velocity u_1 is much smaller than the sound speed c_1.
- If there is some perturbation, then sound will occur and carry the gas to the front again.
- The front is then disturbed by those gas carried by the sound.

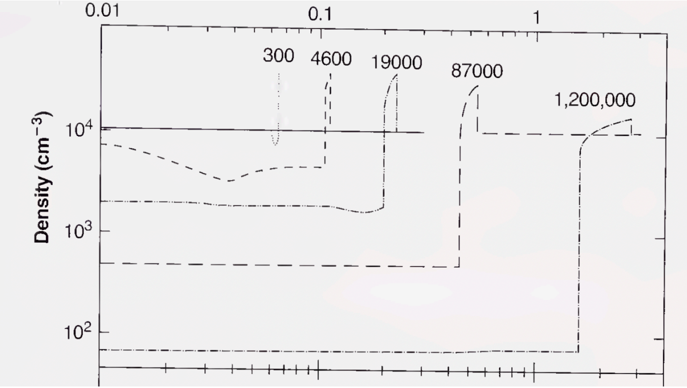
As a consequence, the weak R front is the leading shock at initial growth.

6.5 Time evolution of HII region

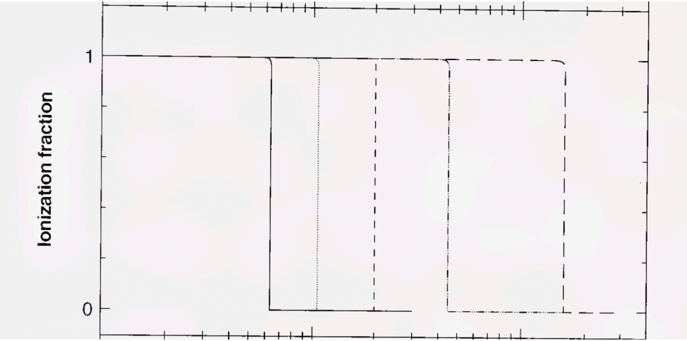
Whole picture of the time evolution.

- The weak R front is the leading shock at initial growth. Many ionizing photons exist behind the front, and the front moves very fast.
- As the front moves, $r \nearrow$ and $\phi \searrow$, then $u_0 \searrow u_R$. From this time onward, the R front can no longer exist.
- Then, a shock front breaks off from the ionization front and D ionization front follows the shock.
- The shock front gradually weakens and only the strong D front remains. And the D front moves slowly and expands the HII region.

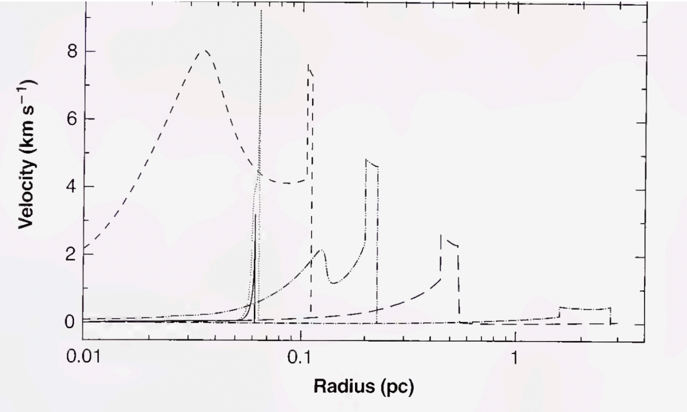
6.5 Numerical calculation of expanding HII region



6.5 Numerical calculation of expanding HII region

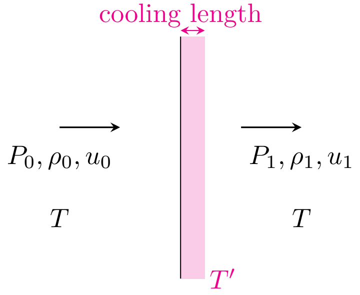


6.5 Numerical calculation of expanding HII region



6.5 Additional comments on expanding HII region

- During the D phase the gas motions within the HII region become important. To achieve the equilibrium pressure, gas velocity inside the HII region varies with radius (see fig6.1).
- Near the ionization front, temperature becomes higher because of the advection of H^0 gas. This result in relatively strong emission lines.
- cooling length: The thickness of the layer in which the electron temperature rises to its peak and then falls to the equilibrium temperature.

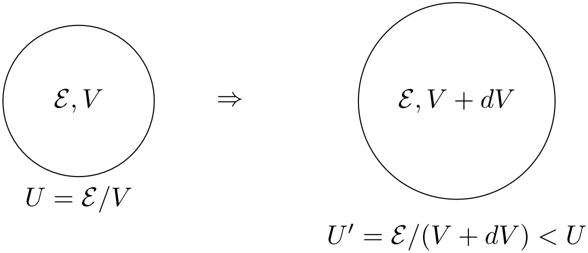


Answer to Quiz!

Quiz!

The energy equation of U (eq(16)) has the dilation term $-U\nabla\cdot\boldsymbol{u}$ on its RHS. However, the fundamental form (eq(24)) doesn't have a term like $-E\nabla\cdot\boldsymbol{u}$. **Why ?**

U: The internal energy per unit volume
Here we denote total energy $\mathcal{E}$ of a certain fluid element.

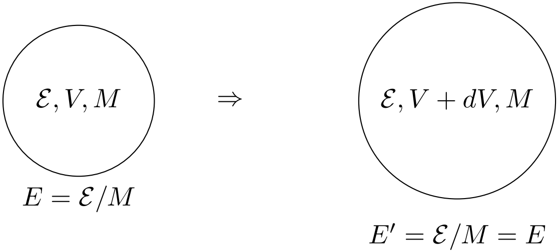


Therefore, we need the dilation term $-U\nabla\cdot\boldsymbol{u}$.

Answer to Quiz!

$E = U/\rho$: The internal energy per unit mass.
We denote total energy $\mathcal{E}$, total mass M.

$$E = \frac{U}{\rho} = \frac{\mathcal{E}/V}{M/V} = \frac{\mathcal{E}}{M} \tag{99}$$



The effect of the expansion $V + dV$ is canceled out ($\because$ both U and ρ decrease).