On the Mathematical Formulation of Empirical Laws

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Examples of Scaling Relations

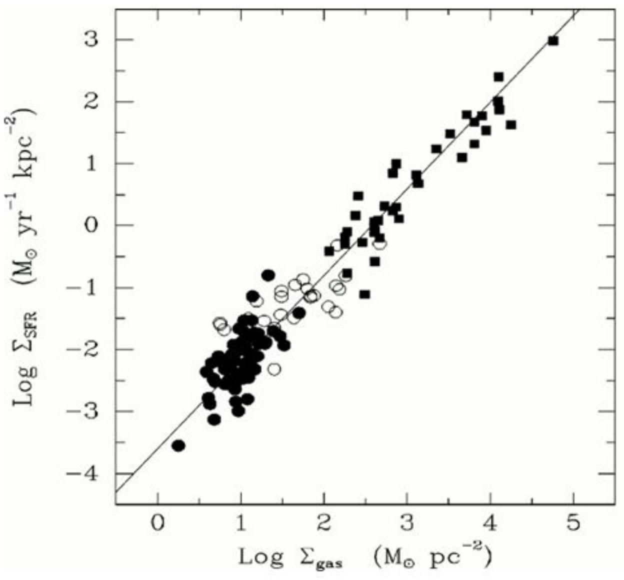
Kennicutt-Schmidt Law:

$$\frac{\dot{\Sigma}_{SF}}{\dot{\Sigma}_{SF0}} = \left(\frac{\Sigma_{gas}}{\Sigma_0}\right)^{1.4}$$

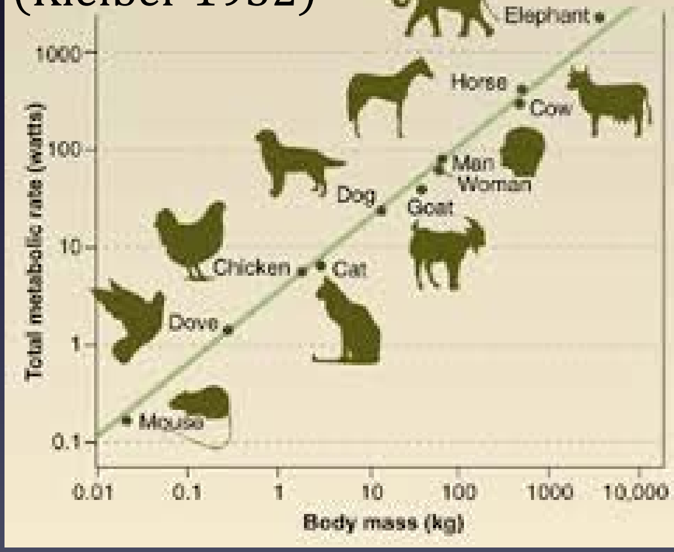
Kleiber's Law:

$$\frac{\dot{B}_{MR}}{\dot{B}_0} = \left(\frac{M}{M_0}\right)^{0.75}$$

(Kennicutt 1998)



(Kleiber 1932)



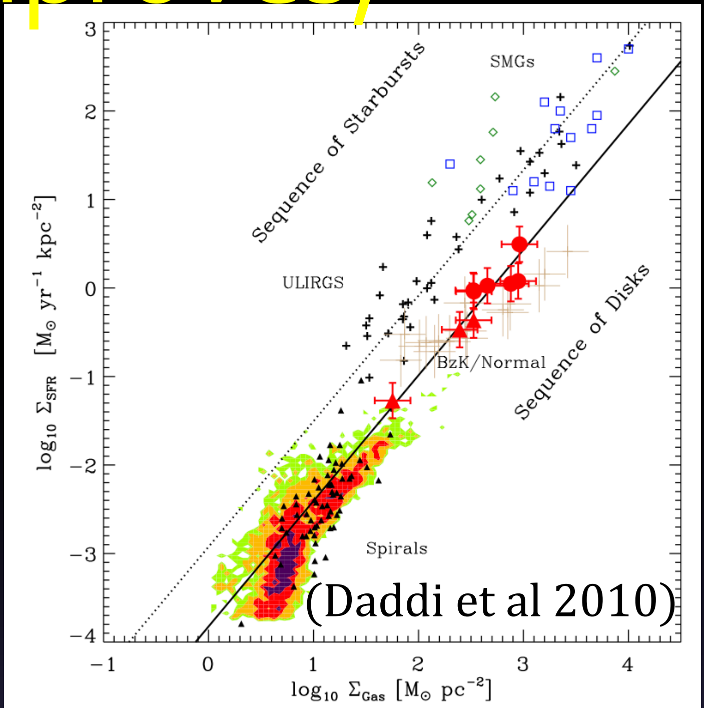
However as datasets improves, differences appear…

Kennicutt-Schmidt Law:

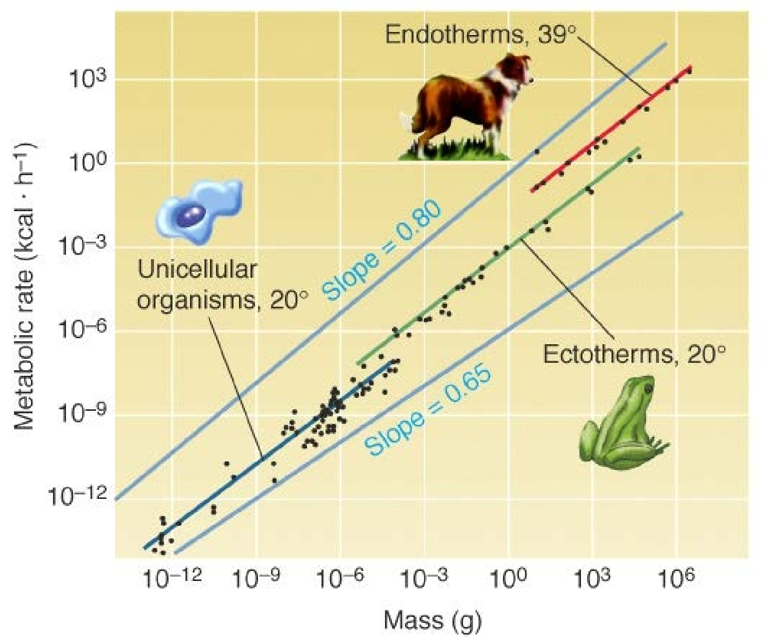
$$\frac{\dot{\Sigma}_{SF}}{\dot{\Sigma}_{SF0}} = \left(\frac{\Sigma_{gas}}{\Sigma_0}\right)^{1.4}$$

***Kleiber's Law*:**

$$\frac{\dot{B}_{MR}}{\dot{B}_0} = \left(\frac{M}{M_0}\right)^{0.75}$$



(Daddi et al 2010)



Evidence for secondary parameters or something beyond that?

In the Kennicutt-Schmidt Law:

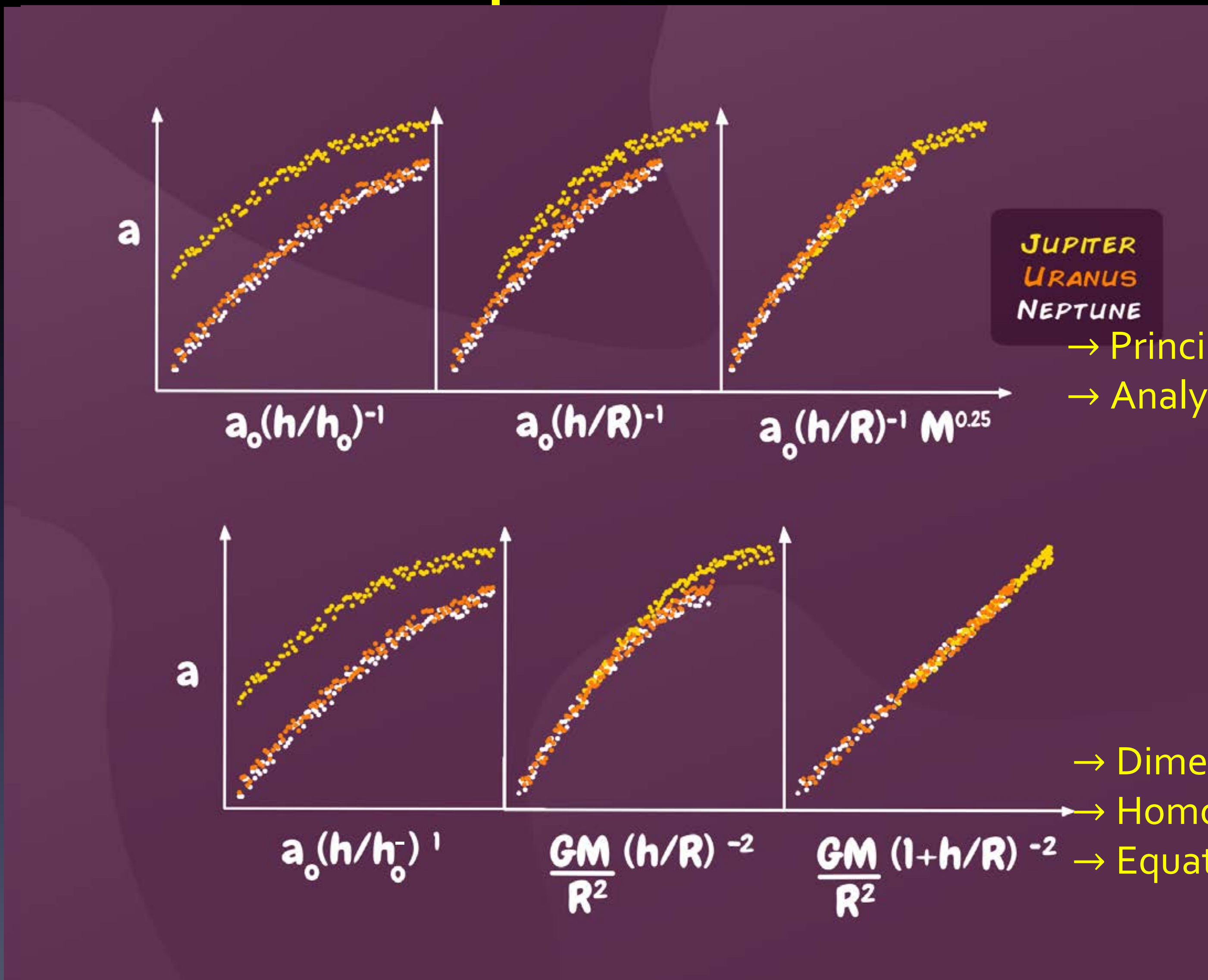
$$\frac{\dot{\Sigma}_{SF}}{\dot{\Sigma}_{SF0}} = \left(\frac{\Sigma_{gas}}{\Sigma_0}\right)^{1.4} \longrightarrow \dot{\Sigma}_{SF} = f\ (\Sigma_{gas}, \Omega, f_{H2}, \mathcal{M}, \alpha_{CO}, \Sigma_{star}, P_{turb}, etc)$$

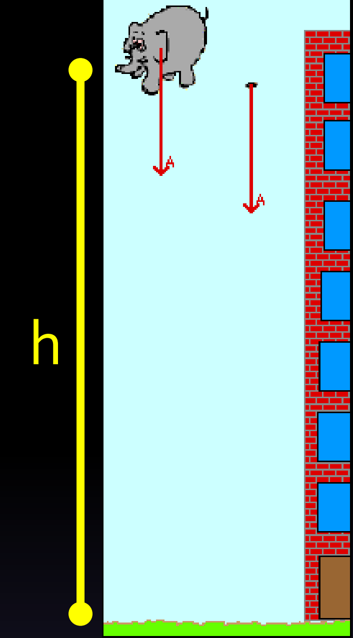
$$\frac{\dot{\Sigma}_{SF}}{\dot{\Sigma}_{SF0}} = \left(\frac{\Sigma_{gas}}{\Sigma_0}\right)^{1.4} \left(\frac{\Omega}{\Omega_0}\right)^{0.6} \ldots \text{(Principal Component Analysis)}$$

However, mathematically speaking a relation that requires as many constants as variables CANNOT BE a meaningful LAW of nature (Bridgman 1922), since laws must be independent of the units employed to measure the variables (Fourier 1822, Théorie de la Chaleur).

Only **homogeneous equations** in their various units of measurement fulfil this requirement, for example: $\dot{\Sigma}_{SF} = \varepsilon_0\, \Sigma_{gas}\, \Omega$

Free Fall Acceleration Experiments





→ Principal Component
→ Analysis

→ Dimensionally
→ Homogeneous
→ Equation

Educated Guess: π theorem

$F(A_1, A_2, \ldots, A_n) = 0 \rightarrow f(\pi_1, \pi_2, \ldots, \pi_{n-k}) = 0$; For k=3 (mass, length and time):

In the **FREE FALL example**, if n=5 (a, G, M, R, h) $\rightarrow a = \frac{GM}{R^2} f(h/R)$

In **STAR FORMATION LAW**:

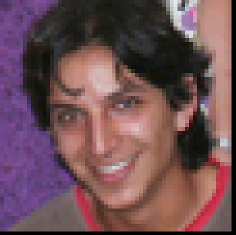
if n=4 $(\dot{\Sigma}_{SF}, \Sigma_{gas}, v, L) \rightarrow \dot{\Sigma}_{SF} = \varepsilon_o \Sigma_{gas} v L^{-1} = \varepsilon_o \Sigma_{gas} \Omega$ (Silk 1997; Elmegreen 1997)

if n=4 $(\dot{\Sigma}_{SF}, \Sigma_{gas}, G, L) \rightarrow \dot{\Sigma}_{SF} = \varepsilon_o \sqrt{\frac{G}{L}} \Sigma_{gas}^{1.5}$ (Corrected K-S; Escala 2015)

if n=5 $(+\Omega) \rightarrow \dot{\Sigma}_{SF} = \varepsilon \left(\Omega / \underbrace{\sqrt{G\Sigma_{gas}/L}}_{t_{ff}^{-1}}\right) \sqrt{\frac{G}{L}} \Sigma_{gas}^{3/2}$

....Etc

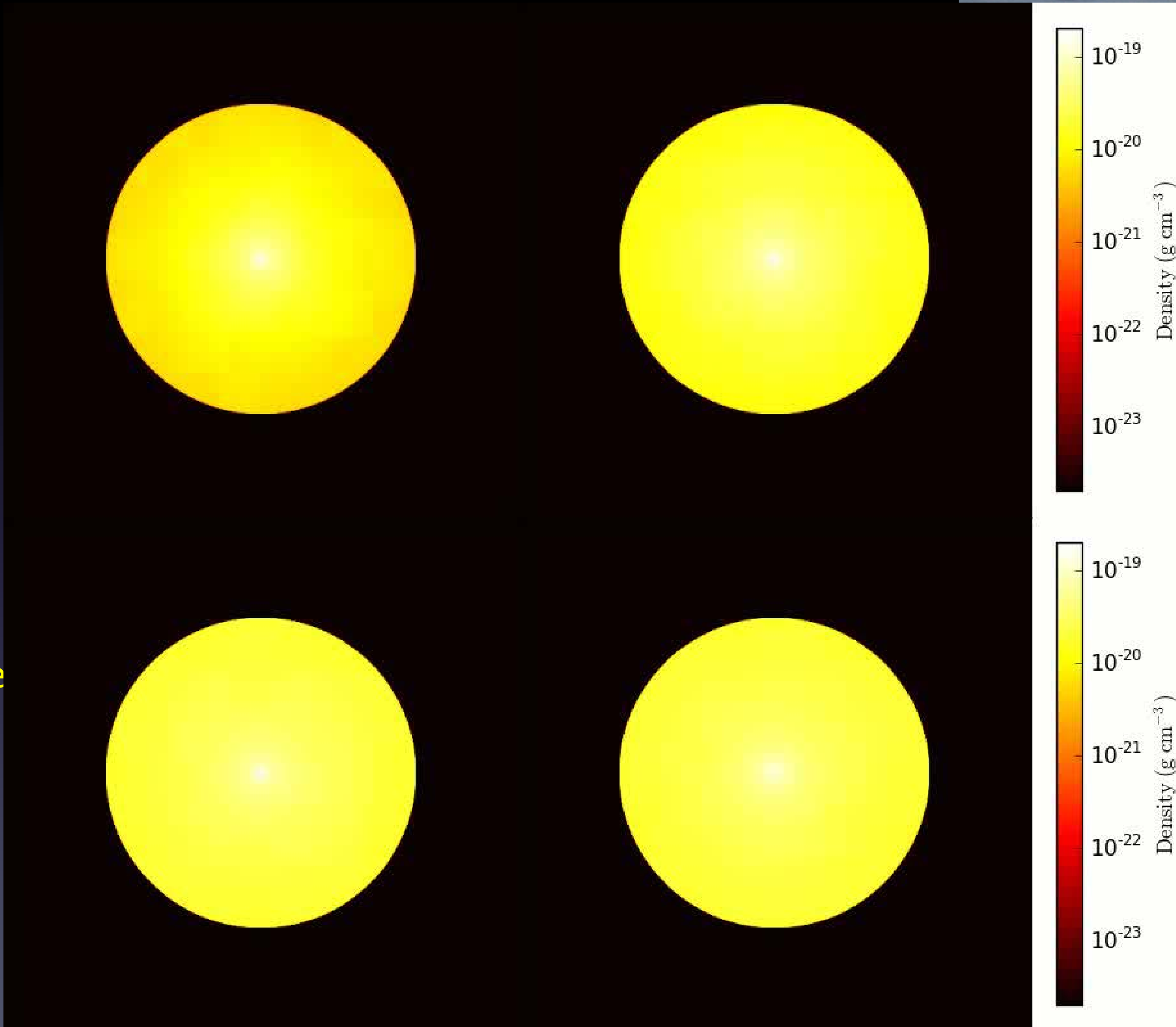
Star Formation Laws



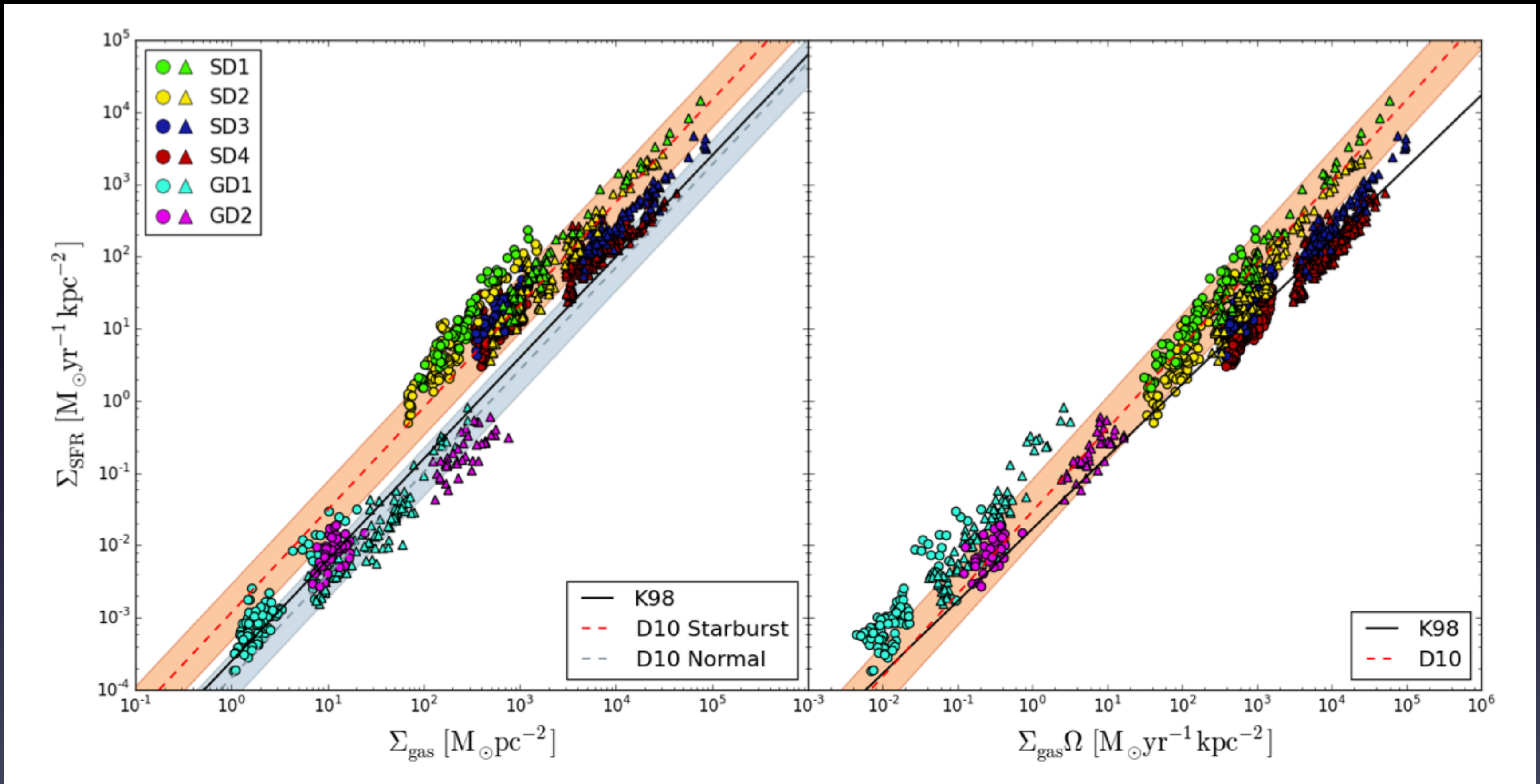
Numerical Experiments

Galactic Disk Simulations using ENZO (Adaptive Mesh Refinement), for both Spiral and Starburst Disks.

To isolate the the effects of galactic rotation, we run the SAME gas configuration, only varying Ω thru the galactic potential.

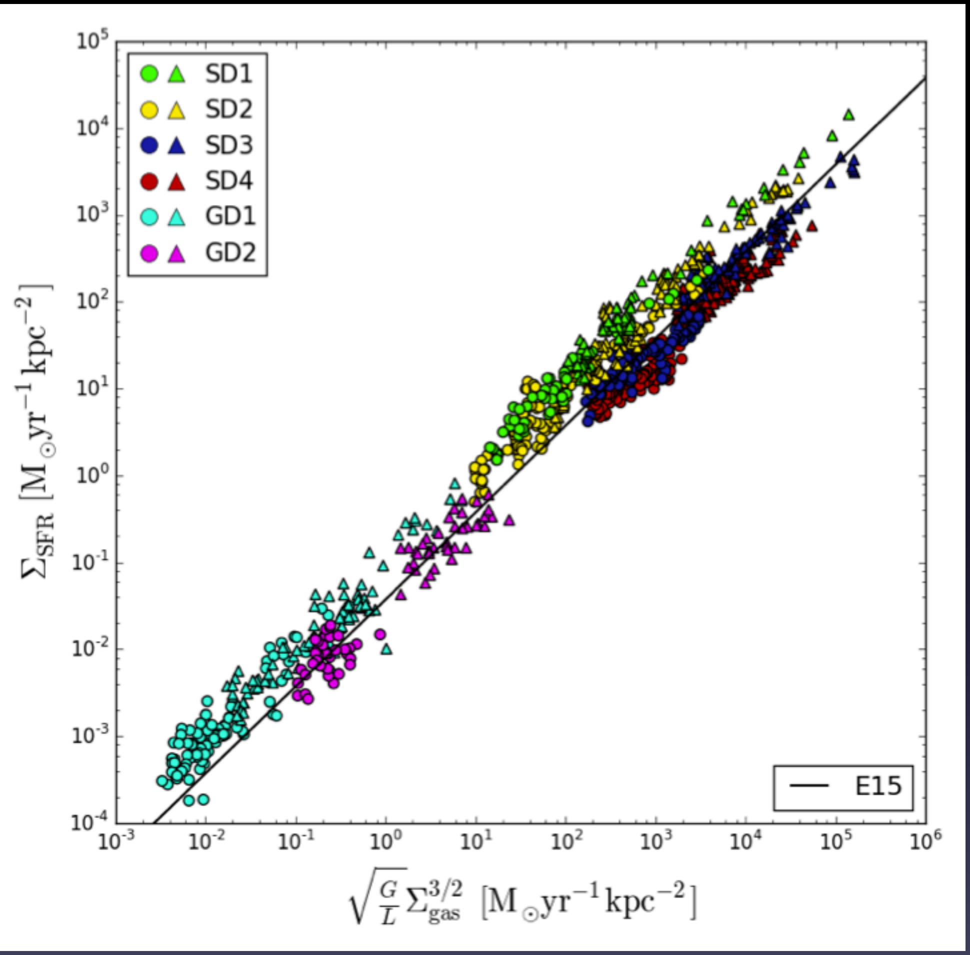


Kennicutt-Schmidt & Silk-Elmegreen Relations:

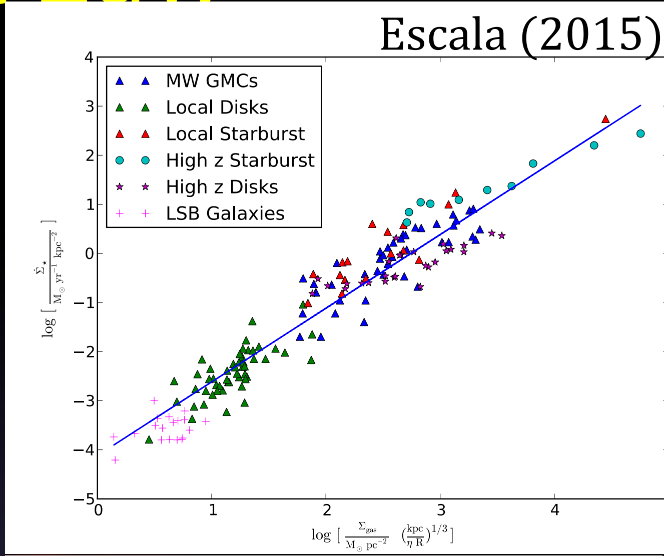


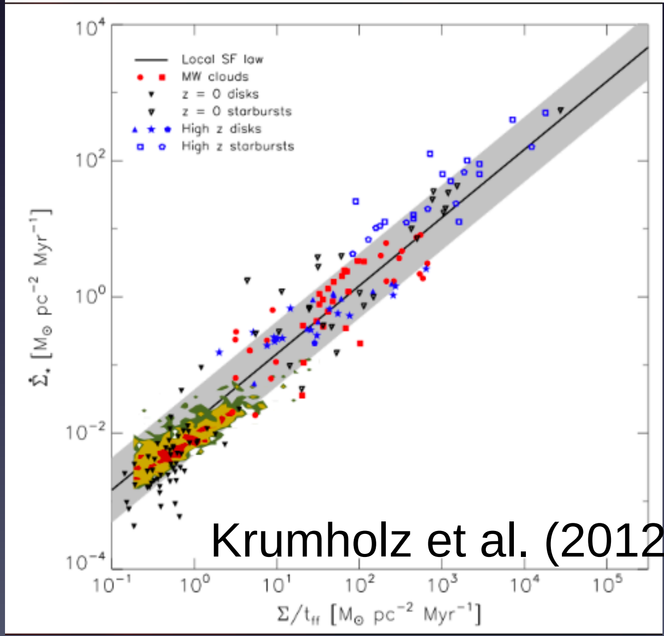
Utreras, Becerra & Escala (2016)

Dimensionally Corrected Kennicutt-Schmidt Law



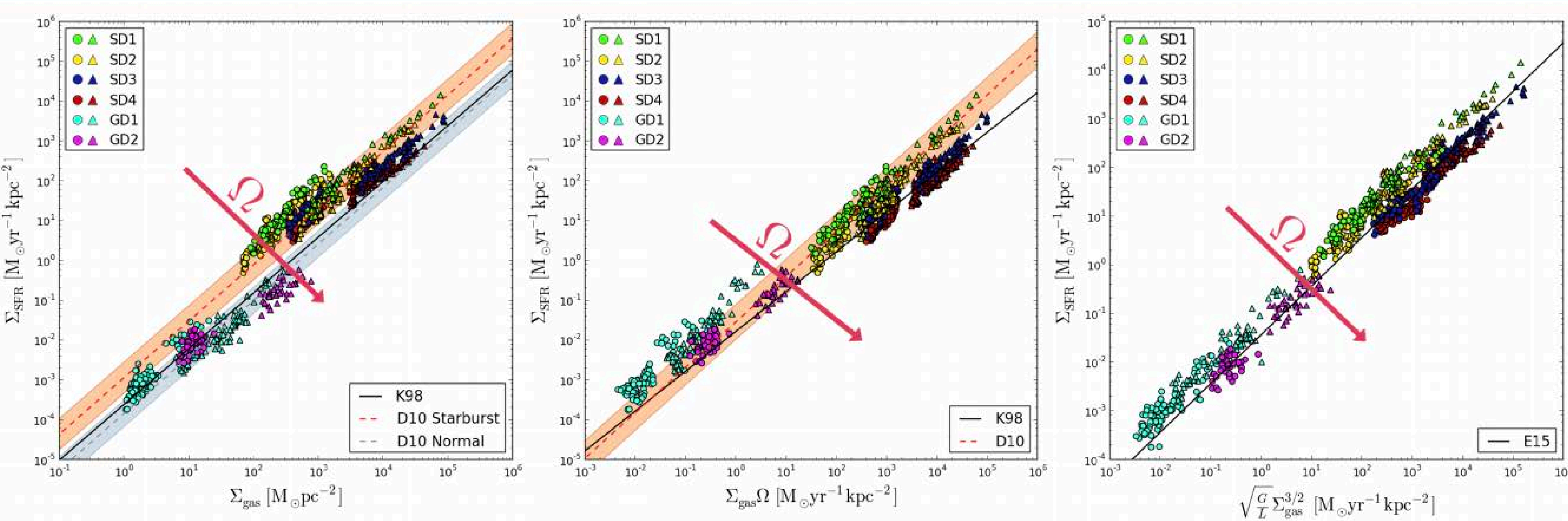
Utreras, Becerra & Escala (2016)





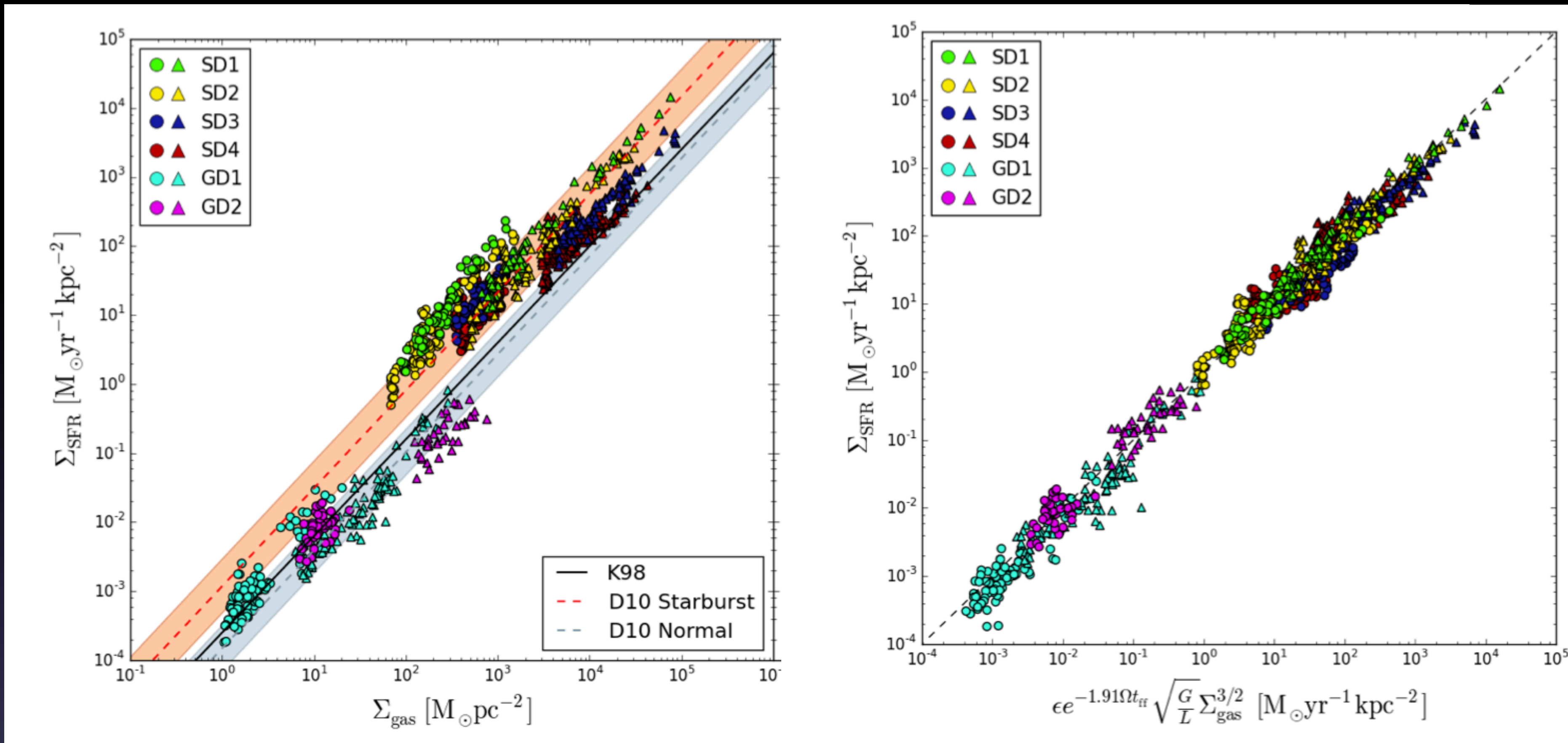
L=length integration in the Line of Sight

Ω in the Star Formation Relations



- Kennicutt (1998) - Schmidt (1959)
- Silk (1997) - Elmegreen (1997)
- Escala (2015)

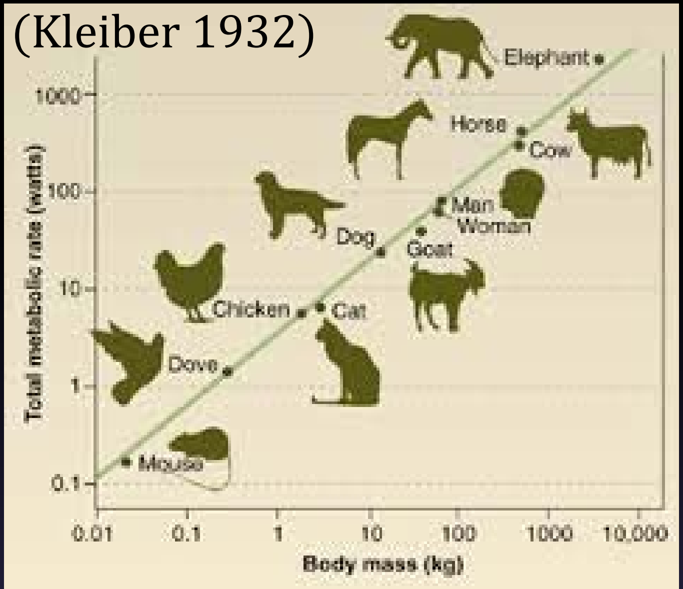
Kennicutt-Schmidt vs this work



Star formation Relation	Scatter [dex]
KS	0.490
Bi-modal KS	0.360
SE	0.362
E15	0.316
This work	0.206

Metabolic Rate Relation

The “Fire” of Life



Kleiber’s Law:

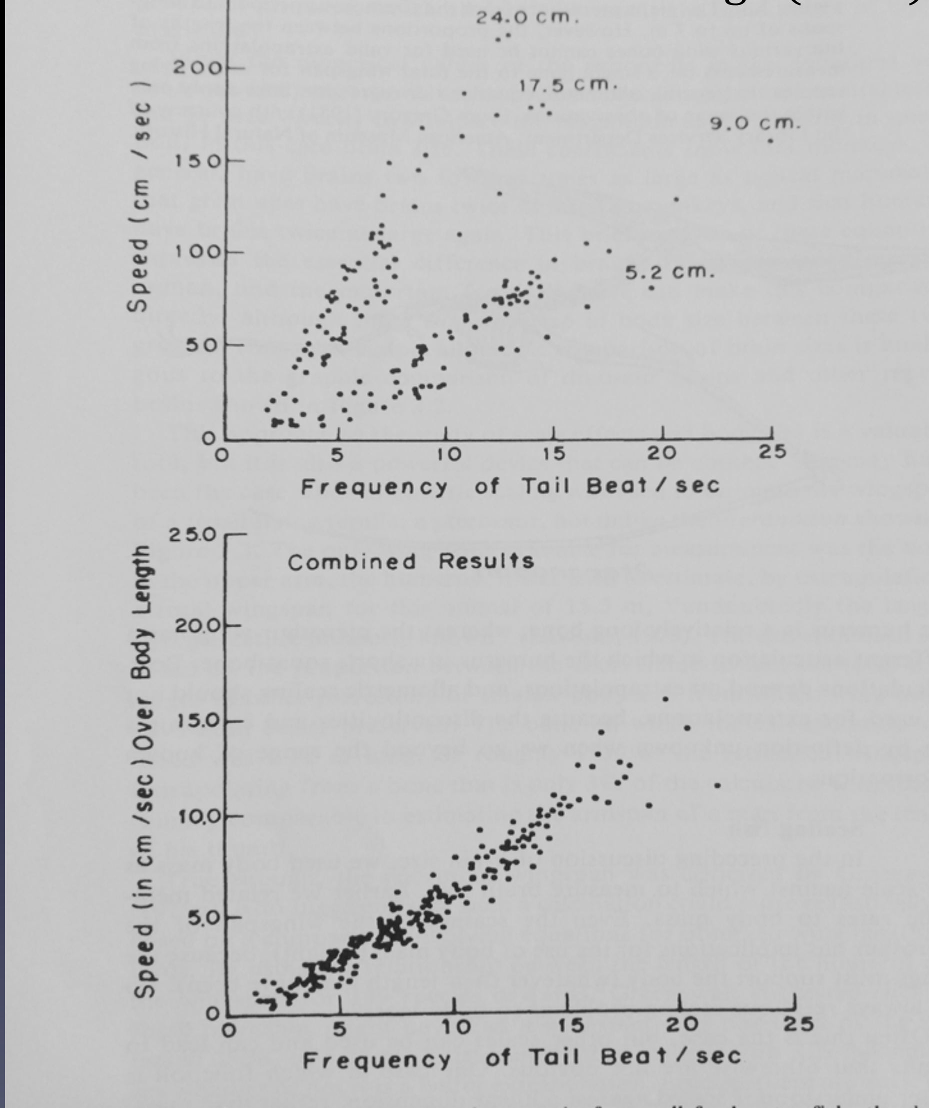
$$\frac{\dot{B}_{MR}}{\dot{B}_0} = \left(\frac{M}{M_0}\right)^{3/4}$$

- Basal metabolic rate (energy consumption under resting conditions) as a function of the animal’s mass.
- It’s scales as ¾, instead of the 2/3 expected for surface energy losses

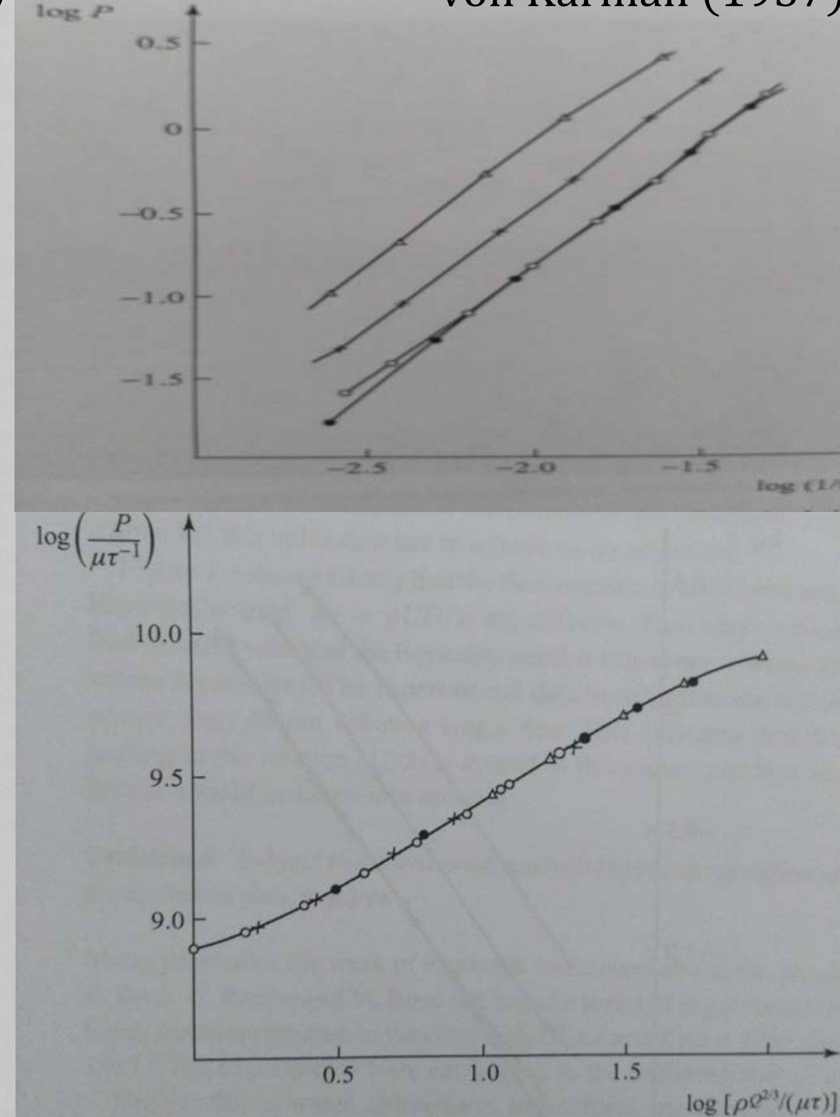
in isometric bodies: $\dot{B}_{MR} \propto S \propto L^2 \propto \left(\sqrt[3]{\frac{M}{\rho_0}}\right)^2 \propto M^{2/3}$

It is possible to fulfil dimensional homogeneity in Biology?

Brainbridge (1958)

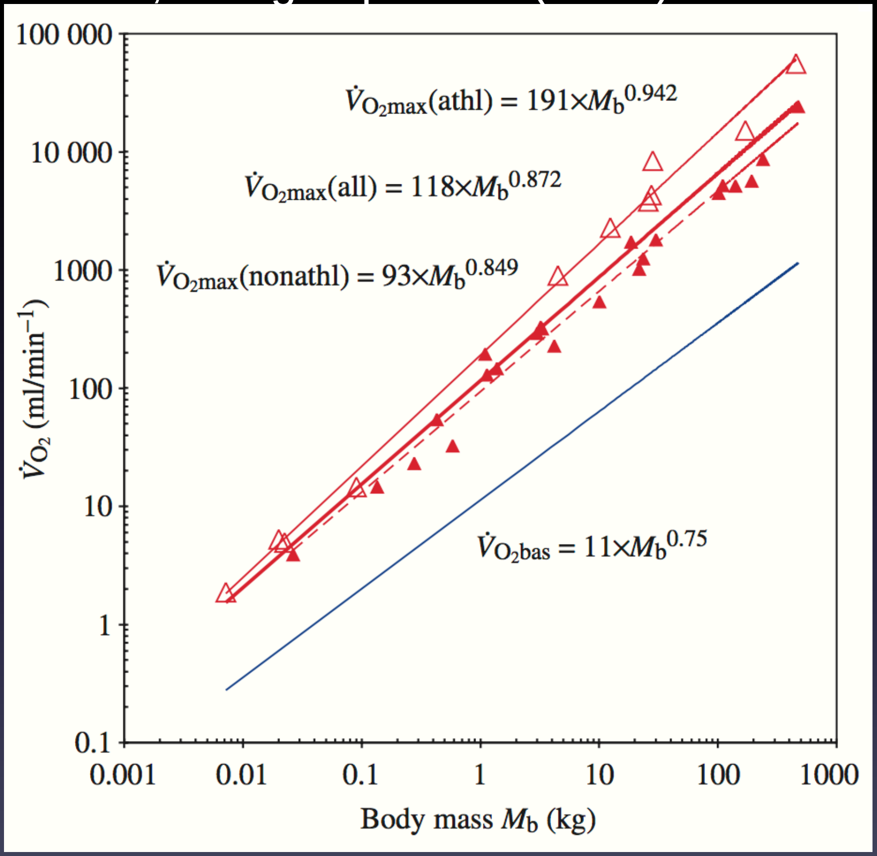


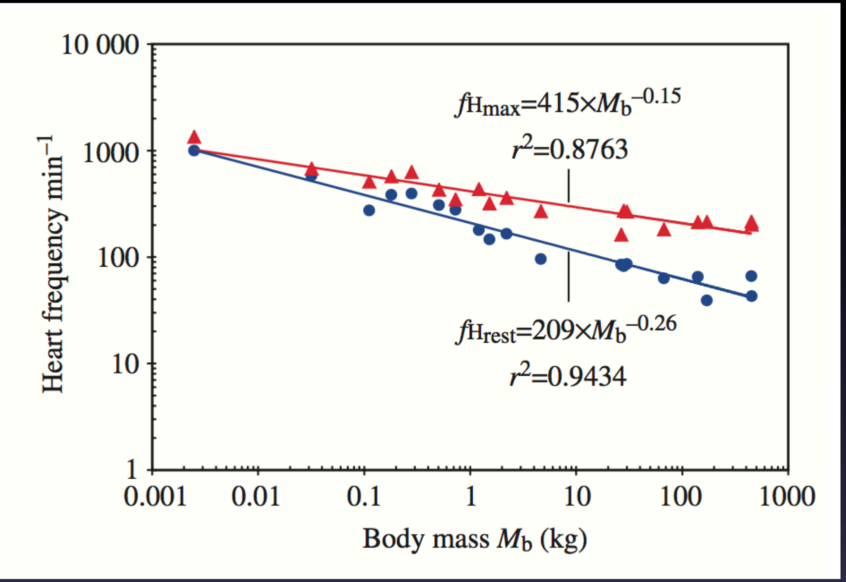
von Kármán (1957)



Metabolic Rate in Running vs Resting Mammals

Weibel, Bacigalupe et al (2004)





Weibel & Hoppeler (2005)

Different allometric exponents but $\frac{\dot{V}_{O_2}}{f_H} \propto M^{1.0}$

π Theorem

$F(A_1, A_2, ..., A_n) = 0 \rightarrow f(\pi_1, \pi_2, ..., \pi_{n-k}) = 0$; For k=3 (kg, mlO_2 & sec):

In the **METABOLIC RATE**:

if n=4 ($\dot{V}_{O2}$, f_H, η_{O2}, W) $\rightarrow \dot{V}_{O2} = \varepsilon_o \, \eta_{O2} \, f_H \, W$

if n=6 (+ T, T_a) $\rightarrow \dot{V}_{O2} = \varepsilon(T/T_a)\eta_{O2} \, f_H \, W$

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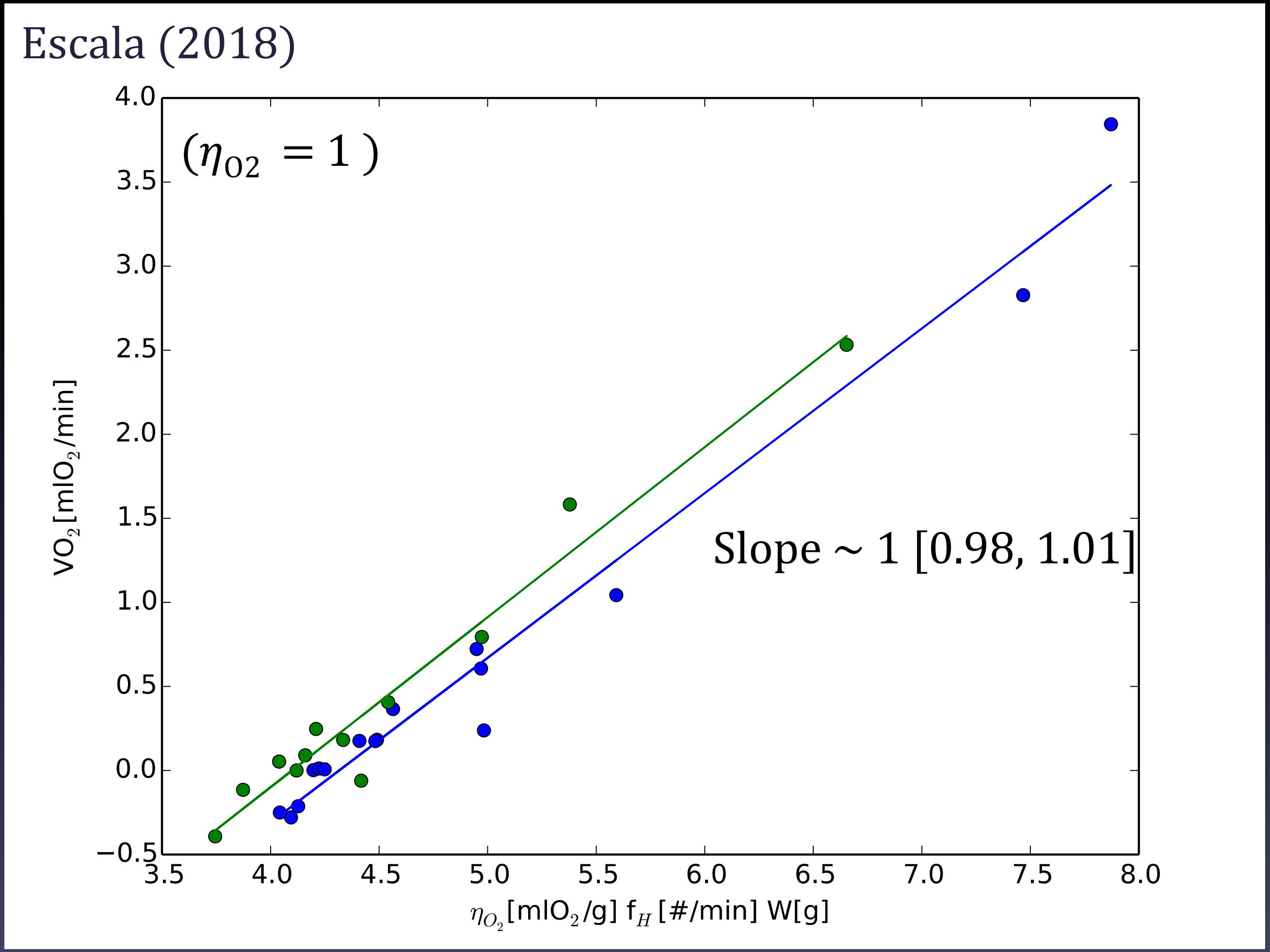
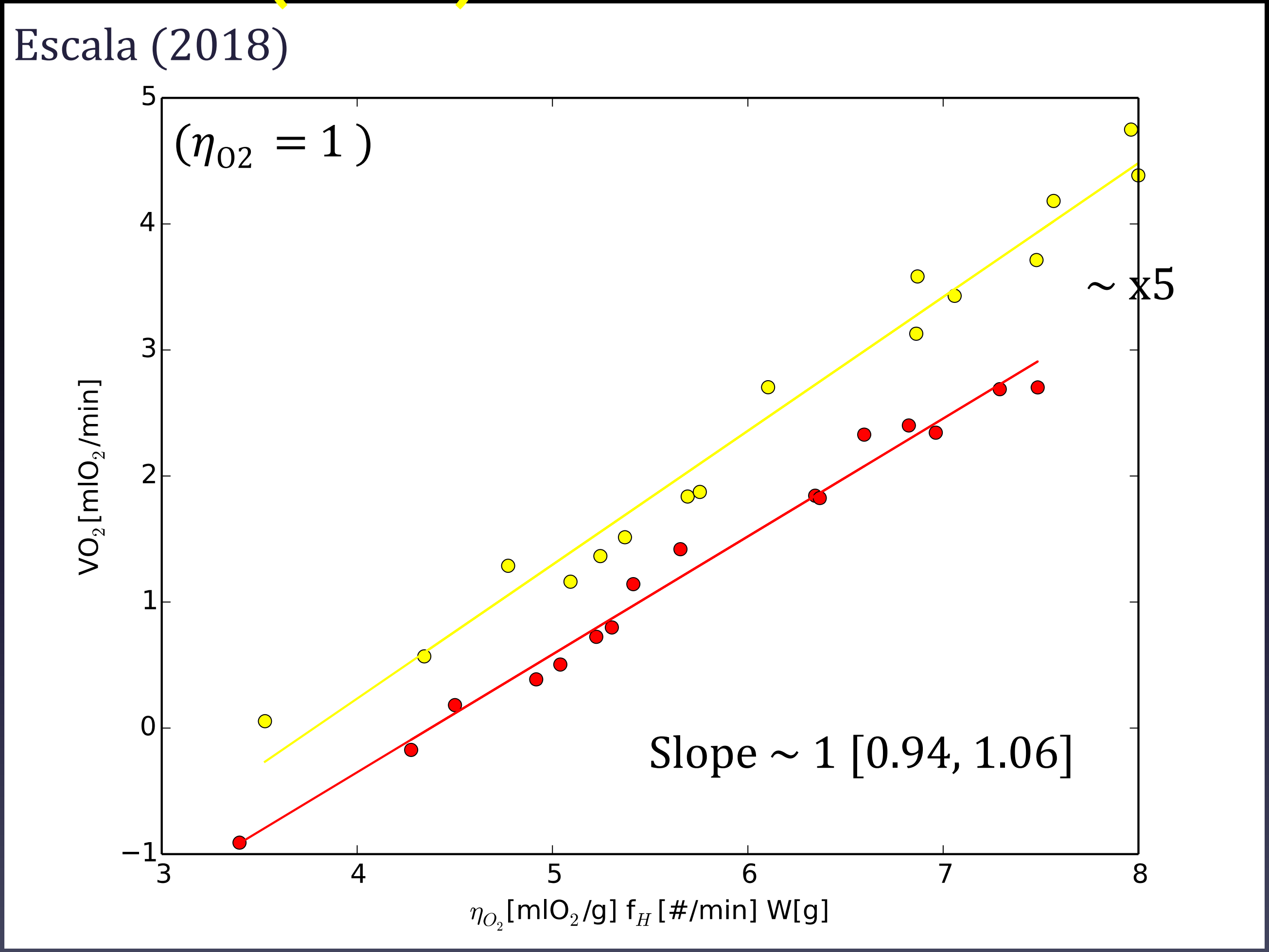
$\dot{V}_{O2}$ in Birds(green) and Mammals(blue)



$$\dot{V}_{O2} = \varepsilon_o \, \eta_{O2} \, f_H \, W$$

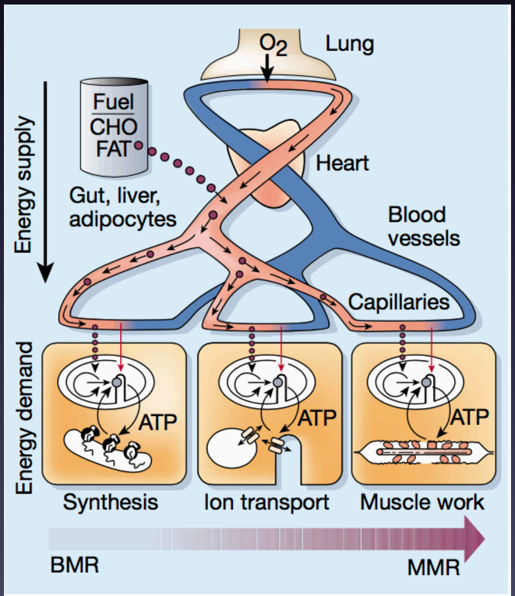
$\dot{V}_{O2}$ in Running (yellow) vs Resting (red) Mammals



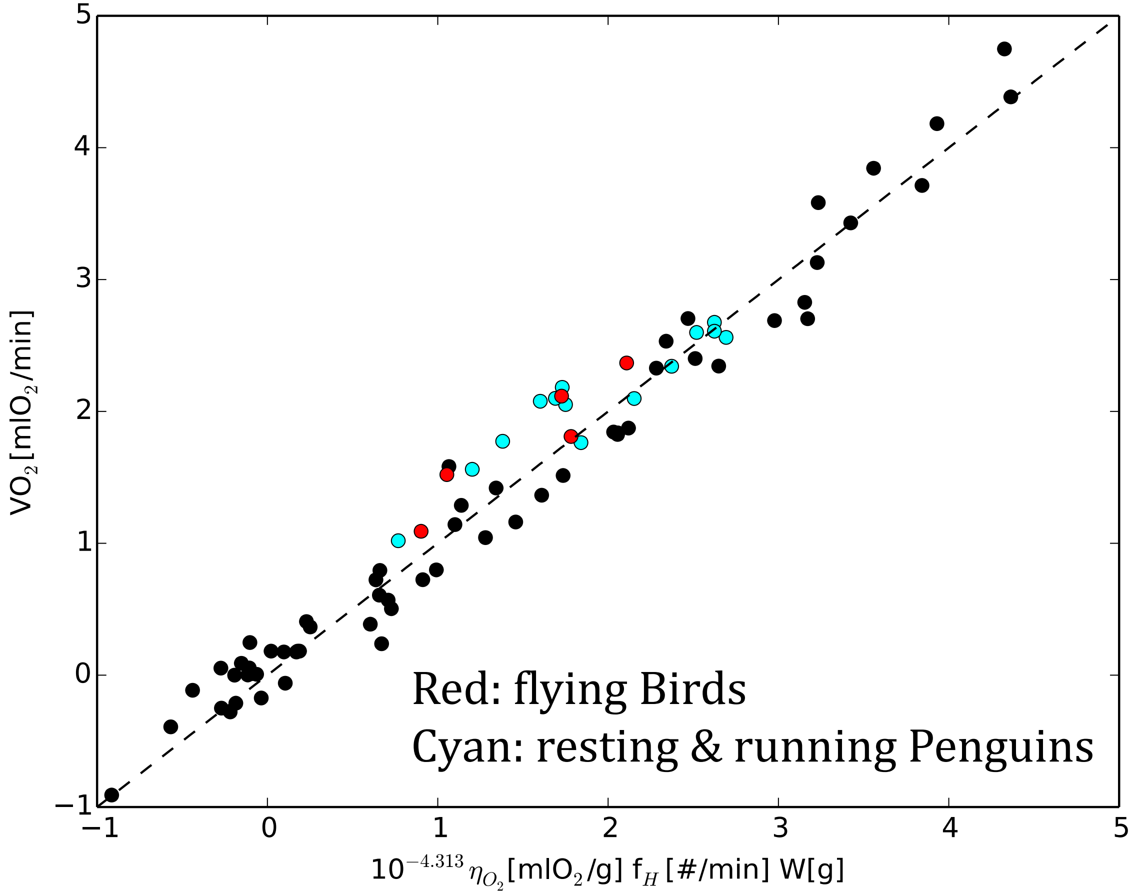
$$\dot{V}_{O2} = \varepsilon_o \, \eta_{O2} \, f_H \, W$$

Unique homogeneous equation for the metabolic rates $\dot{V}_{O2}$

$\eta_{O2}{}^{max} \sim 5\,\eta_{O2}{}^{rest}$



Escala (2018)



Number of Constants with Dimensions

ALLOMETRY (sub sub area of Biology)

$$\frac{\dot{B}_{MR}}{\dot{B}_0} = \left(\frac{M}{M_0}\right)^{0.75}$$

$$\frac{\dot{B}_{MAX}}{\dot{B}_{MAX0}} = \left(\frac{M}{M_0'}\right)^{0.85}$$

$$\frac{f_{MAX}}{f_{MAX0}} = \left(\frac{M}{M_0''}\right)^{-0.15}$$

$$\frac{f_B}{f_{B0}} = \left(\frac{M}{M_0'''}\right)^{-0.25}$$

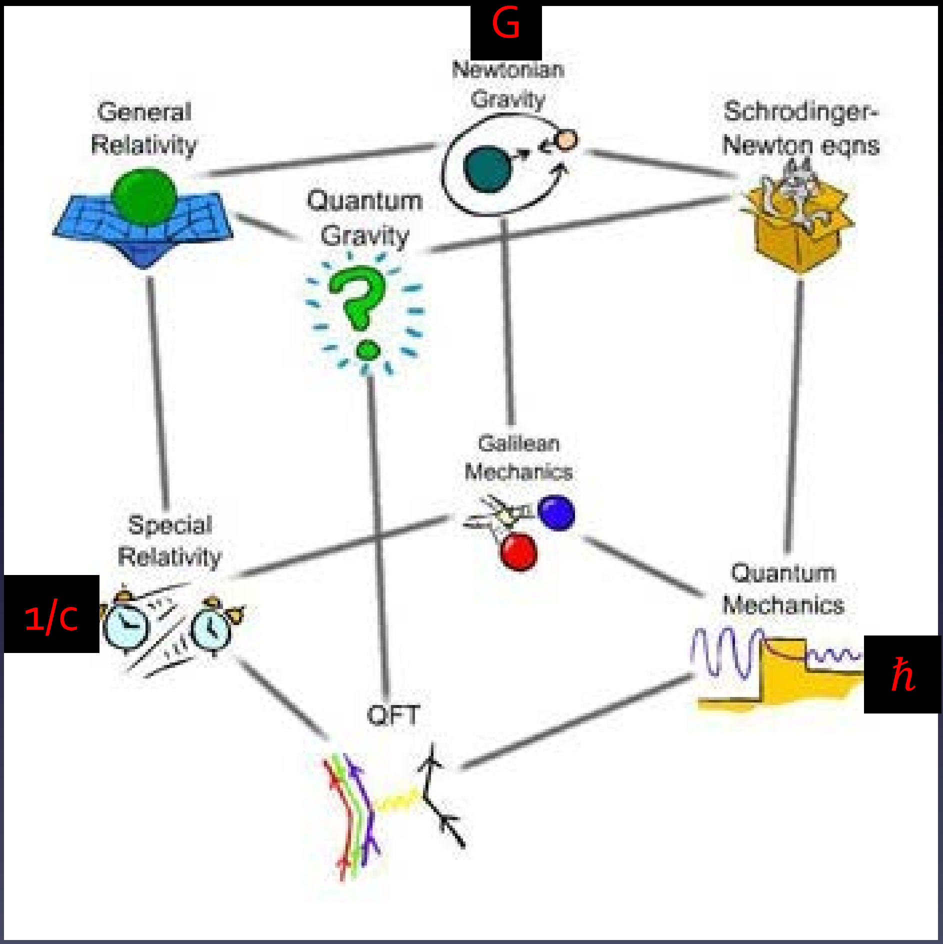
Etc.

$$\epsilon_0 \eta_{O_2} = 10^{-4.313}\ \mathrm{mlO_2 g^{-1}}$$

PHYSICS

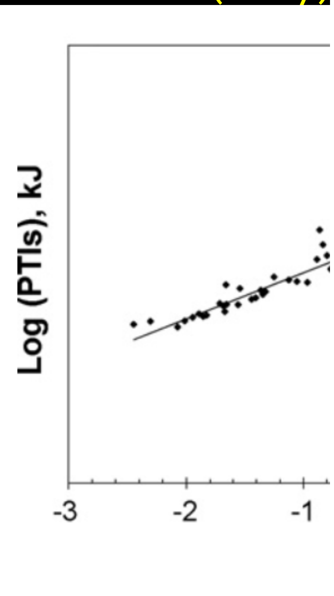
$G, c, \hbar \rightarrow m_p, t_p, l_p$

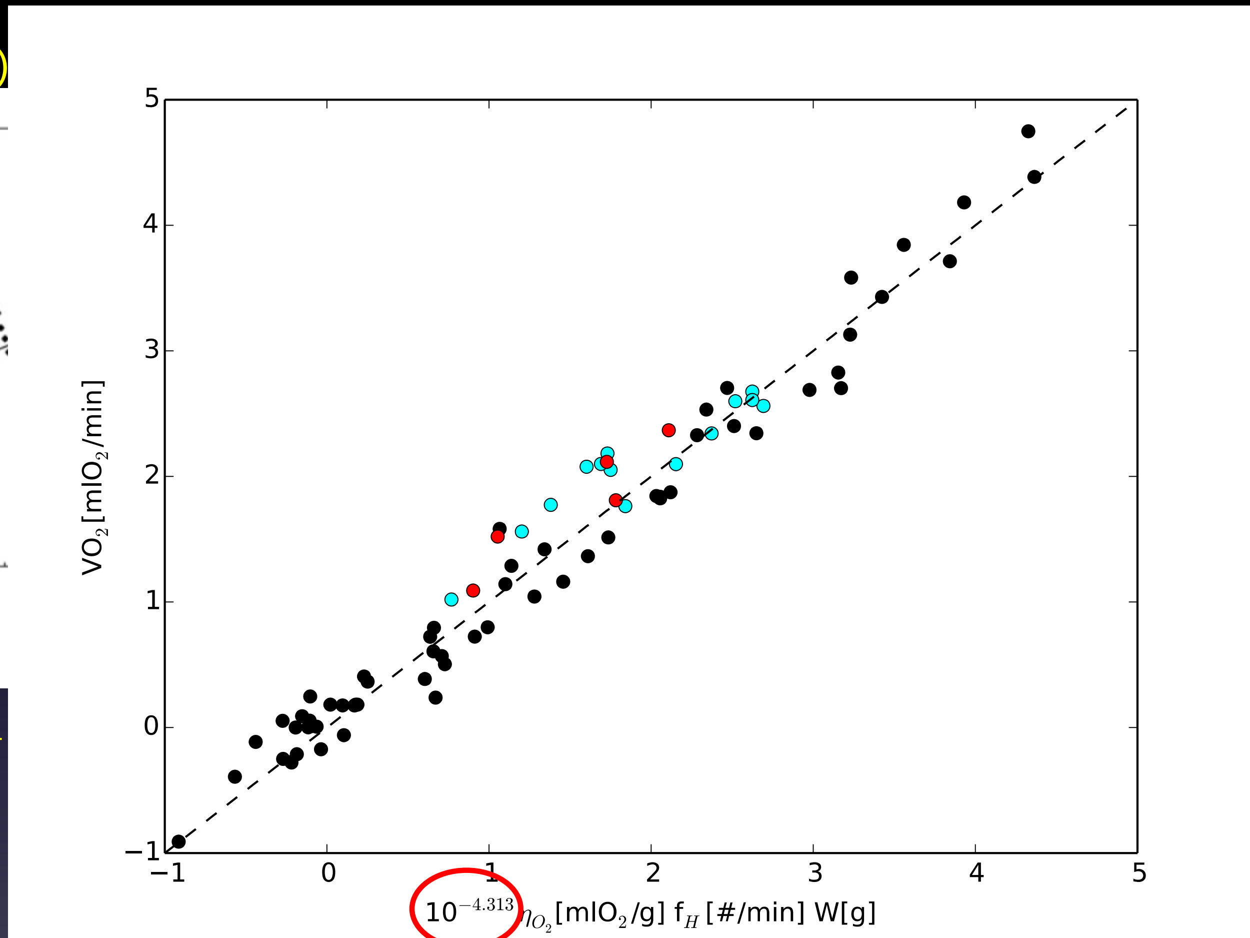
Physical Theories

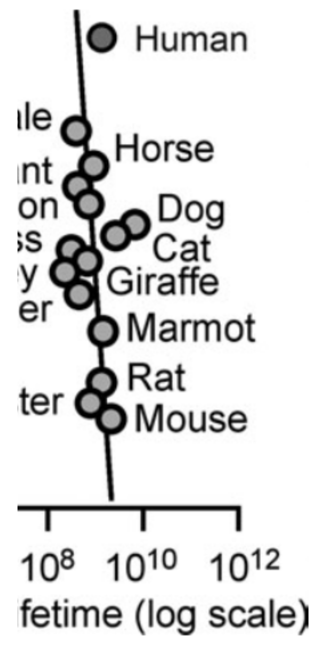


Total Metabolic Energy per life-span & Mass

Atanasov (2007)







$\dot{B}_{MR} T_{ls} = A_{ls}^{+}$

ook et al. 2006)

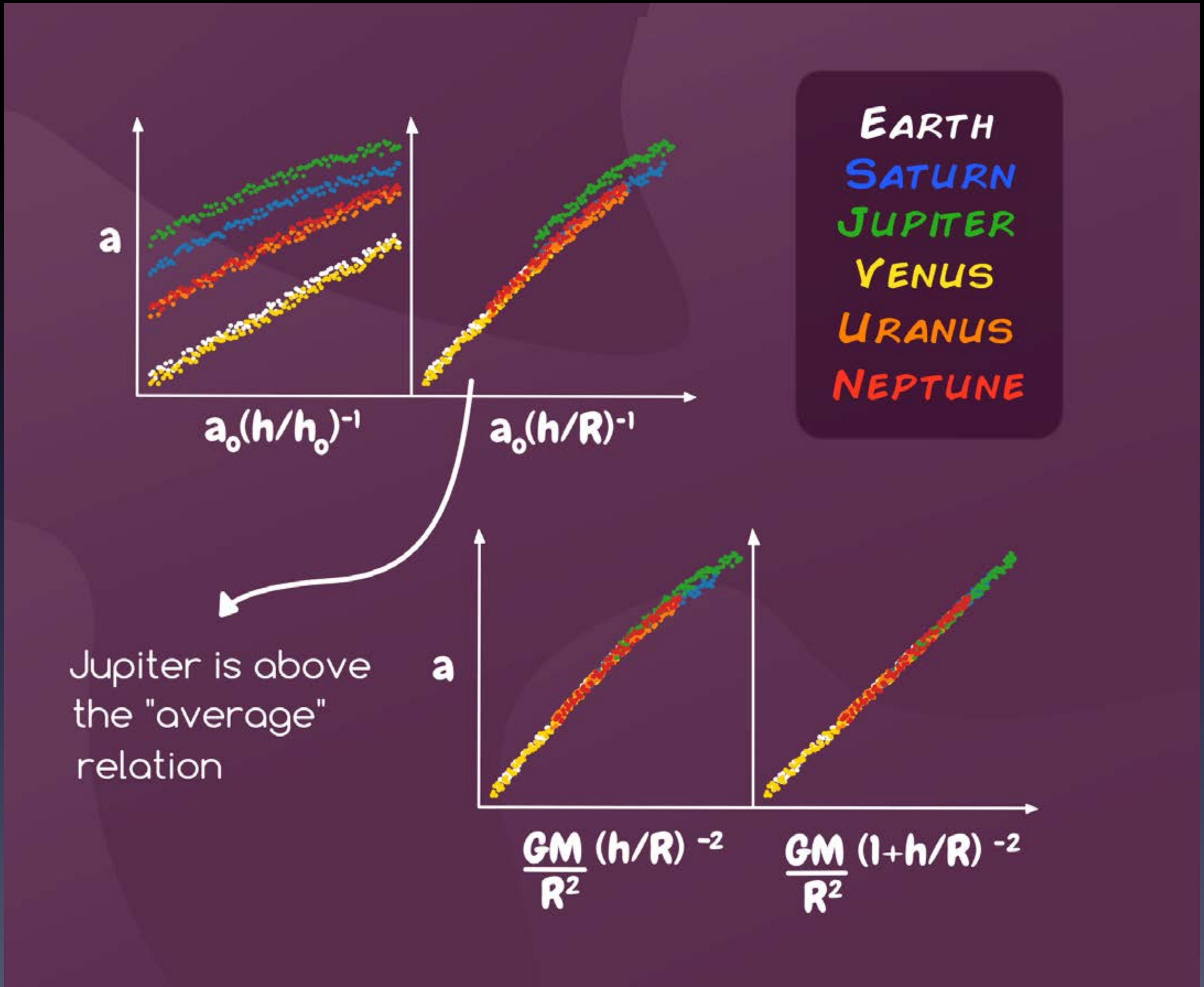
$\rightarrow \; A_{ls}^{+} / N = 10^{-4.45}\, mlO_2 g^{-1} = \varepsilon_0\, \eta_{O2}$

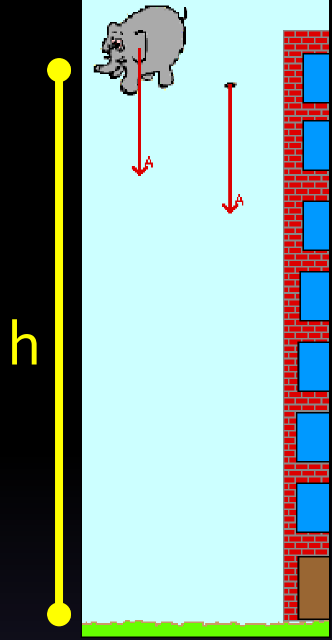
Summary

- Well defined empirical laws must satisfy dimensional homogeneity (Rayleigh's similitude principle).
- We reformulated the star formation law to fulfil this principle and in addition, we use homogeneity to constrain the role of the orbital frequency in the SF efficiency.
- We reformulated the metabolic rate relation, unifying the relation for different classes of animals and aerobic conditions into a single formula.

THANKS!

Free Fall Acceleration Experiments





Confounding Variables Problem

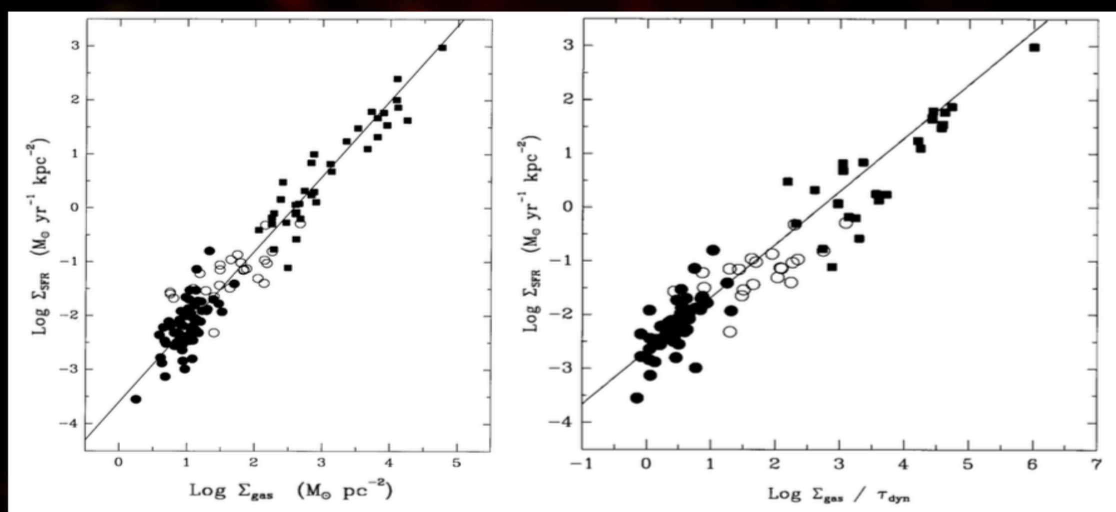
- Several physical processes usually dynamically coupled
- Difficult study of their independent effects

Eg.

$$\Omega = \Omega(M_{\rm gas}, \ldots)$$

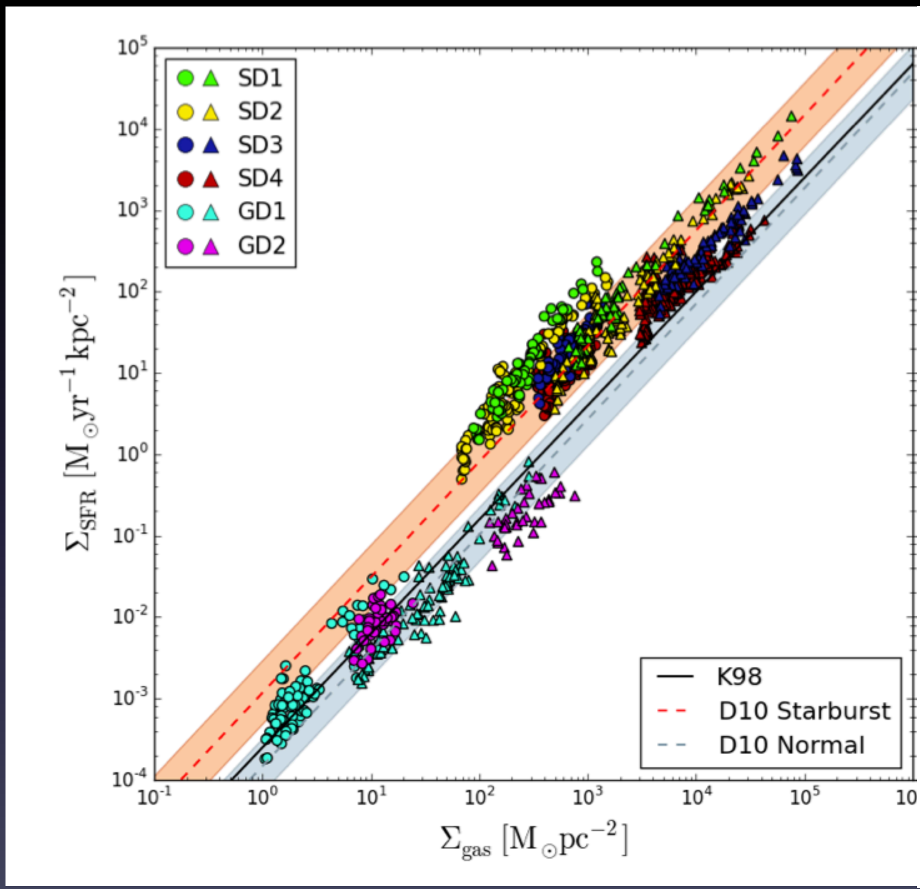
$$t_{\rm ff} = t_{\rm ff}(M_{\rm gas}, \ldots)$$

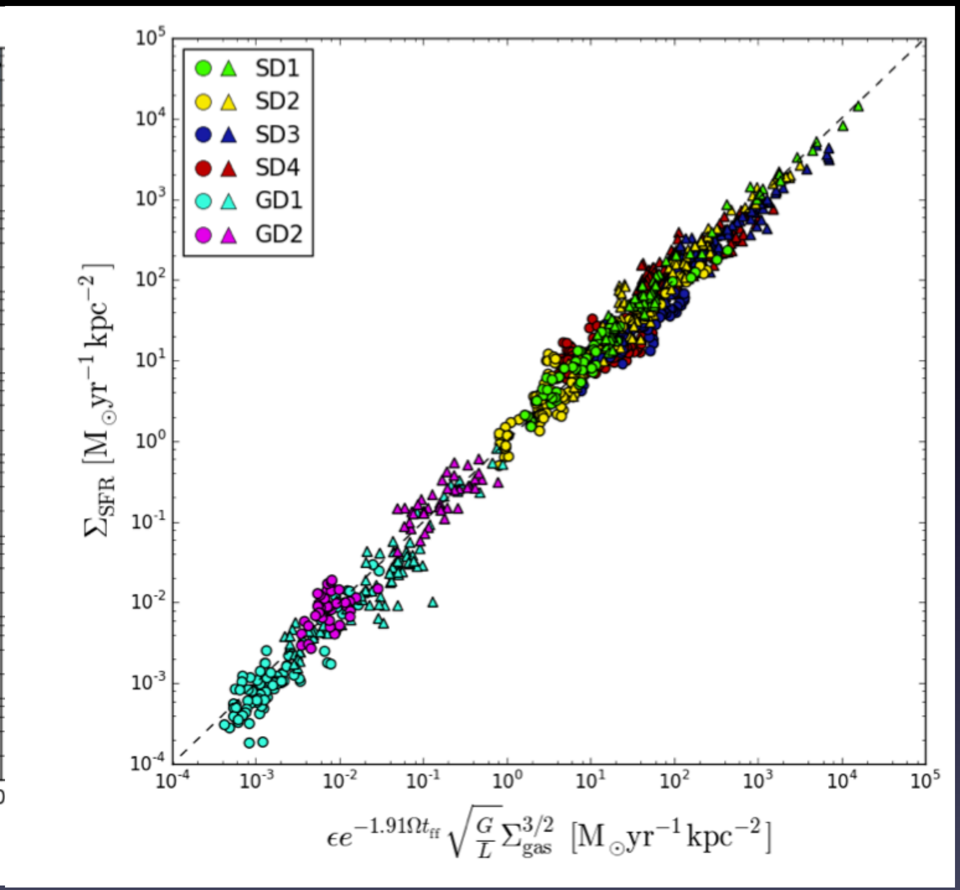
Kennicutt 1998



1. Kennicutt-Schmidt $\Sigma_{\rm SFR} \propto \Sigma_{\rm gas}^{1.4}$ $(\propto \Sigma_{\rm gas}/t_{\rm ff}?)$
2. Silk-Elmegreen $\Sigma_{\rm SFR} \propto \Sigma_{\rm gas}/t_{\rm orb}$

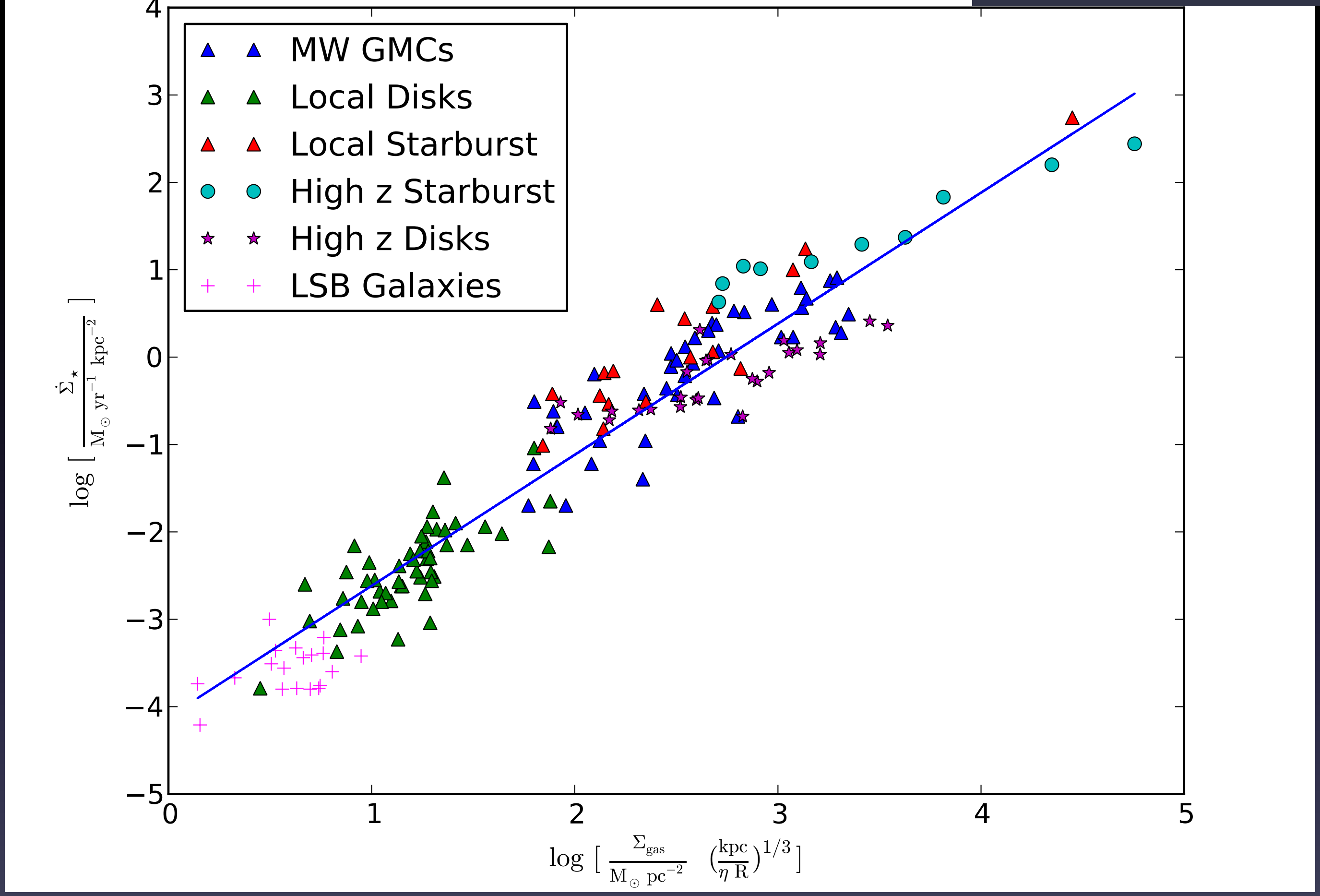
Secondary physical parameters on the Kennicutt-Schmidt law





The SF Law as a Single Function

Escala (2015)



- $\sum_{SFR} = \varepsilon \sum_{gas}{}^{3/2}\ L^{-1/2}$

  L=length integration in LOS= ηR

More sophisticated (n>4) Law: Simulations

- Variations on the efficiency are up to a factor of 10 & parameters going on the efficiency are harder to measure -> cannot be currently constrained by observations.
- A better approach is to study variations on the efficiency by numerical experiments.

Caveat: multivariable function

- Star formation should depend on multiple parameters (not only Σ_{gas}) and a law described as function should have (at least) correct units.

- For example, free fall terminal velocity:

$$V_t = \sqrt{(2mg/\rho A\, Cd)}$$